# The Space of Metric Spectral Triples

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# *Spectral Triples*

***Definition (Spectral Triple (Connes, 89))***

A *spectral triple* $\boxed{(\mathfrak{A}, \mathscr{H}, \mathbb{D})}$ is given by:

- a C*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$,
- a self-adjoint operator $\mathbb{D}$ with compact resolvent defined on a dense subspace $\operatorname{dom}(\mathbb{D})$ of $\mathscr{H}$

such that $\{a \in \mathfrak{A} : a\operatorname{dom}(\mathbb{D}) \subseteq \operatorname{dom}(\mathbb{D}), [\mathbb{D}, a] \text{ is bounded}\}$ is a dense *-subalgebra of $\mathfrak{A}$.

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### *Example: Parametrized Families*

- $(C(M), \Gamma_{L^2}(\Sigma M), D\!\!\!/)$ over a closed spin manifold, *for a given Riemannian metric*,
- $(C^*(\mathbb{Z}^d, \sigma), \ell^2(\mathbb{Z}^d, \mathbb{C}^d), D\!\!\!/_{\mathbb{T}^2})$ over a quantum torus *for a given 2-cocycle* $\sigma$.

Are these continuous in their natural parameters?

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*Example: Approximations with Spectral Triples*

- *Matrix models as approximations of quantum spaces (e.g., fuzzy tori)*
- Inductive limits of spectral triples and their limits,
- Graph approximations of fractals and their limiting spaces

What does limit of spectral triples mean?

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*Is there a natural geometry on the space of metric spectral triples?*

Such a space would allow us to study convergence of geometries themselves, and eventually to construct new spectral triples as limits.

# The Topological Space of Metric Spectral Triples

# *Start with Connes' metric*

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Let $(\mathfrak{A}, \mathscr{H}, \not{D})$ be a spectral triple. We define an analogue of the *Lipschitz seminorm* by setting:

$$\boxed{\mathsf{L}_{\not{D}}(a) := |||[\not{D}, a]|||_{\mathscr{H}}}$$

for all $a \in \mathfrak{A}$ such that $a\operatorname{dom}(\not{D}) \subseteq \operatorname{dom}(\not{D})$.

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$\mathsf{L}_{D\!\!\!/}$ induces the *Connes metric* on the state space by duality:

$$\boxed{\mathsf{mk}_{D\!\!\!/}(\varphi, \psi) := \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{A}, |||[D\!\!\!/, a]|||_{\mathscr{H}} \leqslant 1\right\}}$$

for any two states $\varphi, \psi$ of $\mathfrak{A}$.

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for any two states $\varphi, \psi$ of $\mathfrak{A}$. The Connes metric is the analogue of the classical Monge-Kantorovich metric, which metrizes the weak* topology on space of Radon probability measures.

# *Compact Quantum Metric Spaces*

*Definition (Connes, 89; Rieffel, 98, 99; L., 13,15)*

A *quantum compact metric space* $(\mathfrak{A}, \mathsf{L})$ is a unital C*-algebra $\mathfrak{A}$ and a lower semicontinuous seminorm $\mathsf{L}$ defined on a dense $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$ such that:

1. $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}$,
2. the *Monge-Kantorovich metric* $\mathsf{mk}_{\mathsf{L}}$
$$\mathsf{mk}_{\mathsf{L}} : \varphi, \psi \in \mathscr{S}(\mathfrak{A}) \mapsto \sup\left\{|(\varphi(a) - \psi(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\}$$
metrizes the weak* topology on the state space of $\mathfrak{A}$,
3. $\mathsf{L}(\Re(ab)) \vee \mathsf{L}(\Im(ab)) \leqslant C(\mathsf{L}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}\mathsf{L}(b)) + D\mathsf{L}(a)\mathsf{L}(b)$.

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### *Definition (Rieffel,00; L., 13)*

A *-morphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is a *quantum isometry* from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ when $\mathsf{L}_{\mathfrak{B}}(b) = \inf \mathsf{L}_{\mathfrak{A}}(\pi^{-1}(\{b\}))$. It is full when it is a *-isomorphism whose inverse is a quantum isometry.

# *How Do We Compare Geometries?*

A spectral triple $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is *metric* when the following is a *quantum metric space*

$$(\mathfrak{A}, \mathsf{L}) \text{ where } \mathsf{L}(\cdot) = |||[\mathbb{D}, \cdot]|||_{\mathscr{H}}.$$

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- For compact metric spaces, the natural tool is the *Gromov–Hausdorff distance*.
- It measures how close two metric spaces are up to isometry.

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- For compact metric spaces, the natural tool is the *Gromov–Hausdorff distance*.
- It measures how close two metric spaces are up to isometry.

*We now seek an analogue in the noncommutative setting.*

# *The Gromov-Hausdorff Propinquity*

*Definition (L. 13,14,15)*

If $\pi : (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \to (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a quantum isometry, let

$$\chi(\pi) := \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi^*(\mathscr{S}(\mathfrak{A}))\right).$$

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The *propinquity* between $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is

$$\begin{aligned}\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) := \inf\{\max\{\chi(\pi), \chi(\rho)\} : \\ (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \text{ quantum compact metric space }, \\ \pi : (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \to (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), \rho : (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \text{ quantum iso } \}.\end{aligned}$$

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### *Theorem (L. 13,14,15)*

The propinquity is a *complete metric up to full quantum isometry* on the class of quantum compact metric spaces, whose restriction to classical compact metric spaces is *topologically equivalent to the Gromov-Hausdorff distance*.

# *Extending to Spectral Triples*

If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a metric spectral triple, then

$$(\mathscr{H}, \mathsf{D}_{\not{D}}, \mathfrak{A}, \mathsf{L}_{\not{D}}, \mathbb{C}, 0)$$

where $\mathsf{D}_{\not{D}}$ is the graph norm of $\not{D}$, satisfies the following definition.

*Definition (Metrical C*-correspondences (L.,16))*

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces, and let $(\mathscr{M}, \mathfrak{A}, \mathfrak{B})$ be a C*-correspondence. A *metrical C*-correspondence* $(\mathscr{M}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is given by the additional data of a norm $\mathsf{D}$ on $\mathscr{M}$ such that:

1. $\mathsf{D} \geqslant \left\| \sqrt{\langle \cdot, \cdot \rangle_{\mathscr{M}}} \right\|_{\mathfrak{B}}$,
2. $\{\omega \in \mathrm{dom}(\mathsf{D}) : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $\mathscr{M}$,
3. $\mathsf{L}_{\mathfrak{B}}(\langle \omega, \zeta \rangle_{\mathscr{M}}) \leqslant 2\mathsf{D}(\omega)\mathsf{D}(\zeta)$ for all $\omega, \zeta \in \mathrm{dom}(\mathsf{D})$,
4. $\mathsf{D}(a\omega) \leqslant (\mathsf{L}_{\mathfrak{A}}(a) + \|a\|_{\mathfrak{A}})\mathsf{D}(\omega)$ for all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, $\omega \in \mathrm{dom}(\mathsf{D})$.

# *Quantum Isometries of metrical C*-correspondences*

## *Quantum Isometry*

$$\begin{tikzcd}
(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[r, "\curvearrowright"] \arrow[d, "\pi"'] & (\mathscr{P}, \mathsf{T}) \arrow[d, "\Psi"] \arrow[r, shift right, "\langle\cdot,\cdot\rangle_{\mathscr{P}}"'] & (\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}) \arrow[l, shift right, "\curvearrowleft"'] \arrow[d, "\theta"] \\
(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \arrow[r, "\curvearrowright"] & (\mathscr{M}, \mathsf{D}) \arrow[r, shift right, "\langle\cdot,\cdot\rangle_{\mathscr{M}}"'] & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \arrow[l, shift right, "\curvearrowleft"']
\end{tikzcd}$$

- $\Psi$ is a $\mathbb{C}$-linear surjection, $\pi$, $\theta$ quantum isometries,
- $\Psi(a\omega b) = \pi(a)\Psi(\omega)\theta(b) \quad \forall a \in \mathfrak{A}\ \forall \omega \in \mathscr{P}\ \forall b \in \mathfrak{B}$,
- $\theta(\langle \omega, \zeta \rangle_{\mathscr{P}}) = \langle \Psi(\omega), \Psi(\zeta) \rangle_{\mathscr{M}} \quad \forall \omega, \zeta \in \mathscr{P}$
- $\Psi(\mathrm{dom}(\mathsf{T})) = \mathrm{dom}(\mathsf{D})$,
- $\mathsf{D}(\omega) = \inf \mathsf{T}(\Psi^{-1}(\{\omega\}))$ for all $\omega \in \mathrm{dom}(\mathsf{D})$,

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- $\Psi(\mathrm{dom}\,(\mathsf{T})) = \mathrm{dom}\,(\mathsf{D})$,
- $\mathsf{D}(\omega) = \inf \mathsf{T}(\Psi^{-1}(\{\omega\}))$ for all $\omega \in \mathrm{dom}\,(\mathsf{D})$,
- $\boxed{\chi((\Psi, \pi, \theta)) = \max\left\{\chi(\pi), \chi(\theta)\right\}.}$

# *The Spectral Propinquity*

*Definition (The Spectral Propinquity (L., 20, 22))*

The *spectral propinquity* $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1), (\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2))$ is the infimum of $\varepsilon > 0$ such that there exists a tunnel

$$(\mathscr{H}_1, \mathsf{D}_1, \mathfrak{A}_1, \mathsf{L}_{\not{D}_1}, \mathbb{C}, 0) \xleftarrow{(\Psi_1,\pi_1,\theta_1)} (\mathscr{P}, \mathsf{T}, \mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \mathfrak{E}, \mathsf{L}_{\mathfrak{E}}) \xrightarrow{(\Psi_2,\pi_2,\theta_2)} (\mathscr{H}_1, \mathsf{D}_2, \mathfrak{A}_2, \mathsf{L}_{\not{D}_2}, \mathbb{C}, 0)$$

such that $\chi(\pi_1) \vee \chi(\theta_1) \vee \chi(\pi_2) \vee \chi(\theta_2) \leqslant \varepsilon$

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such that $\chi(\pi_1) \vee \chi(\theta_1) \vee \chi(\pi_2) \vee \chi(\theta_2) \leqslant \varepsilon$ and some appropriate measurement of how far the orbits $(\exp(it\mathbb{D}_1))_{t>0}$ and $(\exp(it\mathbb{D}_2))_{t>0}$ is less than $\varepsilon$.

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such that $\chi(\pi_1) \vee \chi(\theta_1) \vee \chi(\pi_2) \vee \chi(\theta_2) \leqslant \varepsilon$ and

$$\mathsf{Haus}_{\mathsf{mk}_{\mathsf{T},[0,\frac{1}{\varepsilon}]}}\left( \left\{ \left(\langle \exp(i t \mathbb{D}_1) \circ \Psi_1(\cdot), \omega\rangle_{\mathscr{P}}\right)_{0\leqslant t \leqslant \frac{1}{\varepsilon}} : \mathsf{T}(\omega) \leqslant 1 \right\}, \left\{ \left(\langle \exp(i t \mathbb{D}_2) \circ \Psi_2(\cdot), \omega\rangle_{\mathscr{P}}\right)_{0\leqslant t \leqslant \frac{1}{\varepsilon}} : \mathsf{T}(\omega) \leqslant 1 \right\} \right) \leqslant \varepsilon$$

where

$$\mathsf{mk}_{\mathsf{T},J}((\varphi_j)_{j\in J}, (\psi)_{j\in J}) := \sup\left\{ |\varphi_j(\omega) - \psi_j(\omega)| : j \in J, \mathsf{T}(\omega) \leqslant 1 \right\}.$$

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**Main Theorem (L., 20)**

The spectral propinquity is a metric on the class of metric spectral triples, up to *unitary equivalence* of spectral triples.

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The spectral propinquity is a metric on the class of metric spectral triples, up to *unitary equivalence* of spectral triples.

Two spectral triples $(\mathfrak{A}, \mathscr{H}, \slashed{D})$ and $(\mathfrak{B}, \mathscr{S}, \slashed{\nabla})$ are *unitary equivalent* when there exists a unitary $U : \mathscr{H} \rightarrow \mathscr{S}$ such that

- $U \mathrm{dom}(\slashed{D}) = \mathrm{dom}(\slashed{\nabla})$,
- $U \slashed{D} U^* = \slashed{\nabla}$,
- $a \in \mathfrak{A} \longmapsto U a U^*$ is a $*$-isomorphism onto $\mathfrak{B}$.

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**MAIN THEOREM (L., 20)**

The spectral propinquity is a metric on the class of metric spectral triples, up to *unitary equivalence* of spectral triples.

By construction,

$$\Lambda^{\mathrm{spec}}((\mathfrak{A},\mathscr{H},\not{D}),(\mathfrak{B},\mathscr{J},\not{\nabla})) \geqslant \Lambda^{*}((\mathfrak{A},\mathsf{L}_{\not{D}}),(\mathfrak{B},\mathsf{L}_{\not{\nabla}})),$$

so convergence for the spectral propinquity implies metric convergence. But it implies a lot more.

## *Continuity of Spectra*

**Main Theorem (L. 22, 26)**

If $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples converging to a metric spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$ for the spectral propinquity, then for any $\Lambda \in (0,\infty) \setminus \mathrm{Sp}(\mathbb{D}_\infty)$, there exists $N \in \mathbb{N}$ such that, if $n \geqslant N$, then

$$\sum_{\substack{\lambda\in\mathrm{Sp}(\mathbb{D}_n)\\ -\Lambda\leqslant\lambda\leqslant\Lambda}} \mathrm{multiplicity}(\lambda, \mathbb{D}_n) = \sum_{\substack{\lambda\in\mathrm{Sp}(\mathbb{D}_\infty)\\ -\Lambda\leqslant\lambda\leqslant\Lambda}} \mathrm{multiplicity}(\lambda, \mathbb{D}_\infty) < \infty,$$

and moreover,

$$\lim_{n\to\infty} \mathrm{Haus}_{\mathbb{C}}(\mathrm{Sp}(\mathbb{D}_n) \cap [-\Lambda, \Lambda], \mathrm{Sp}(\mathbb{D}_\infty) \cap [-\Lambda, \Lambda]) = 0.$$

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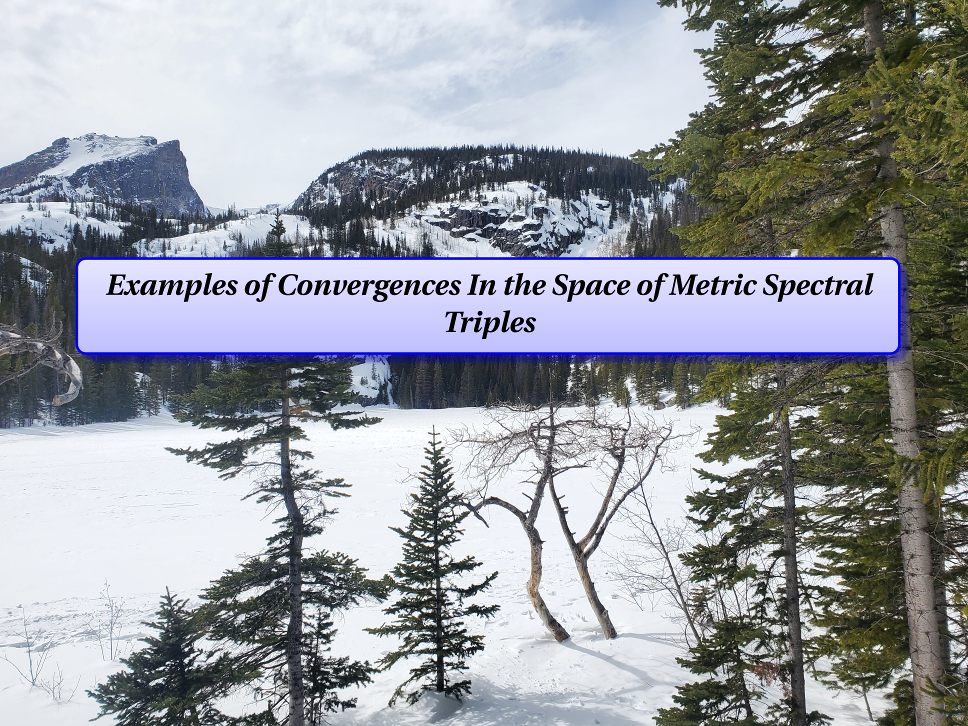
We will say that the spectra converge.

# *Examples of Convergences In the Space of Metric Spectral Triples*



# *Riemannian Spin Geometries*

*Theorem (L. 26)*

Let $M$ be a connected closed spin manifold, and let $g$ be a Riemannian metric over $M$. Endow the space of Riemannian metrics over $M$ with the $C^1$ topology. We then have:

$$\lim_{h\to g} \Lambda^{\mathrm{spec}}((C(M),\Gamma_{L^2}(\Sigma^h M),\slashed{D}_h),(C(M),\Gamma_{L^2}(\Sigma^g M),\slashed{D}_g)) = 0.$$

In particular, the spectrum of $\slashed{D}_h$ converges to the spectrum of $\slashed{D}_g$ as $h$ approaches $g$ in the $C^1$ topology.

# *Almost Commutative Models*

*Theorem (L. 26)*

Let $M$ be a connected closed spin manifold, let $g$ a Riemannian metric over $M$, and let $(\mathfrak{B},\mathscr{F},\mathbb{F})$ be a finite dimensional spectral triple. Endow the space of Riemannian metrics over $M$ with the $C^1$ topology. Endow the space $\mathscr{I}(\mathfrak{B},\mathscr{F})$ of metric spectral triples of the form $(\mathfrak{B},\mathscr{F},\mathbb{G})$ with the topology induced by the operator norm over $\mathscr{F}$. We then have:

$$\lim_{(h,\mathbb{G})\to(g,\mathbb{F})} \Lambda^{\mathrm{spec}}\big((C(M),\Gamma_{L^2}(\Sigma^h M),\slashed{D}_h)\times(\mathfrak{B},\mathscr{G},\mathbb{G}),\\ (C(M),\Gamma_{L^2}(\Sigma^g M),\slashed{D}_g)\times(\mathfrak{B},\mathscr{F},\mathbb{F})\big)=0.$$

In particular, the spectrum of $\slashed{D}_h\times\mathbb{G}$ converges to the spectrum of $\slashed{D}_g\times\mathbb{F}$ as $h$ approaches $g$ in the $C^1$ topology and $\mathbb{G}$ approaches $\mathbb{F}$ in norm.

If $\alpha$ is an ergodic strongly continuous action of a compact Lie group $G$ on a unital C*-algebra, and if $g$ is an inner product on the Lie algebra of $G$, and if

$$\mathbb{D}_g := \sum \alpha^{E_j} \otimes \gamma_j$$

on $L^2(\mathfrak{A}, \mu)$ for $\mu \in \mathscr{S}(\mathfrak{A})$ an $\alpha$-invariant state, and $E_1, \ldots, E_d$ an orthonormal basis of $\mathfrak{G}$, then $(\mathfrak{A}, L^2(\mathfrak{A}, \mu) \otimes \mathbb{C}^d, \mathbb{D}_g)$ is a metric spectral triple (Rieffel, 98).

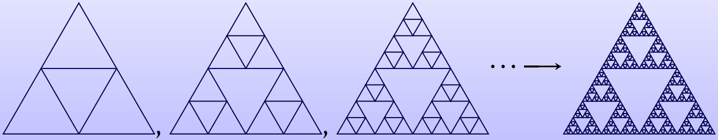
*Theorem (L. 26)*

If $\mathfrak{A}$ is a unital C*-algebra, if $G$ is a compact Lie group, and if $\mathscr{I}$ is the space of all inner products on the Lie algebra $\mathfrak{G}$ of $G$, then

$$g \in \mathscr{I} \longmapsto \left(\mathfrak{A}, \mathscr{H}_g, \mathbb{D}_g\right)$$

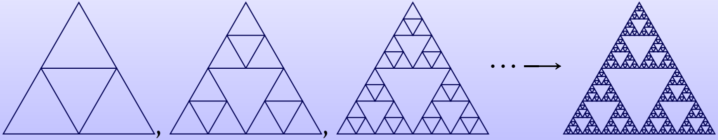
is continuous. In particular, the spectrum of $g \in \mathscr{I} \longmapsto \mathbb{D}_g$ is continuous.

# *Fractal Geometry: Lapidus et al. Convergence*




The Sierpiński gasket is the limit of an increasing sequence of finite graphs. *In what sense?*

# *Fractal Geometry: Lapidus et al. Convergence*



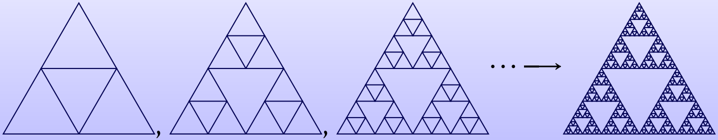
For each $n$, let $\mathscr{S}\mathscr{G}_n$ denote the $n$th graph approximation of the Sierpiński gasket.

The limit

$$\mathscr{S}\mathscr{G}_\infty = \overline{\bigcup_{n\in\mathbb{N}} \mathscr{S}\mathscr{G}_n}$$

is the Sierpiński gasket.

# *Fractal Geometry: Lapidus et al. Convergence*

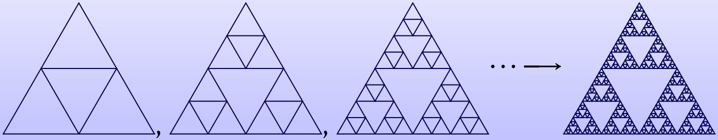


Let $(C([-1,1]), \ell^2([-1,1]), \partial)$ be the natural spectral triple over the interval $[-1,1]$. For each $n \in \mathbb{N} \cup \infty$, let:

- $C(\mathscr{S}\mathscr{G}_n)$ be the C*-algebra of $\mathbb{C}$-valued continuous functions on $\mathscr{S}\mathscr{G}_n$,
- $\mathscr{H}_n = \oplus_{j=1}^{n} \oplus_{k=1}^{3^n} \ell^2([-1,1])$,
- $\mathbb{D}_n = \oplus_{j=1}^{n} \oplus_{k=1}^{3^n} 2^j \partial$

where $f \in C(\mathscr{S}\mathscr{G}_n)$ acts on $(\xi_{j,k})$ by multiplication on each edge.

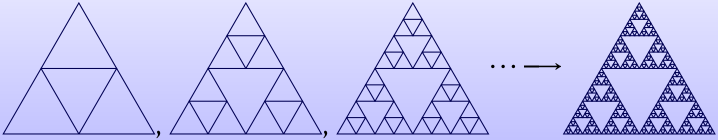
# *Fractal Geometry: Lapidus et al. Convergence*



*Theorem (Landry, Lapidus and L., 2020)*

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}\big((C(\mathscr{S}\mathscr{G}_n),\mathscr{H}_n,\mathbb{D}_n),(C(\mathscr{S}\mathscr{G}_\infty),\mathscr{H}_\infty,\mathbb{D}_\infty)\big) = 0.$$

# *Fractal Geometry: Lapidus et al. Convergence*



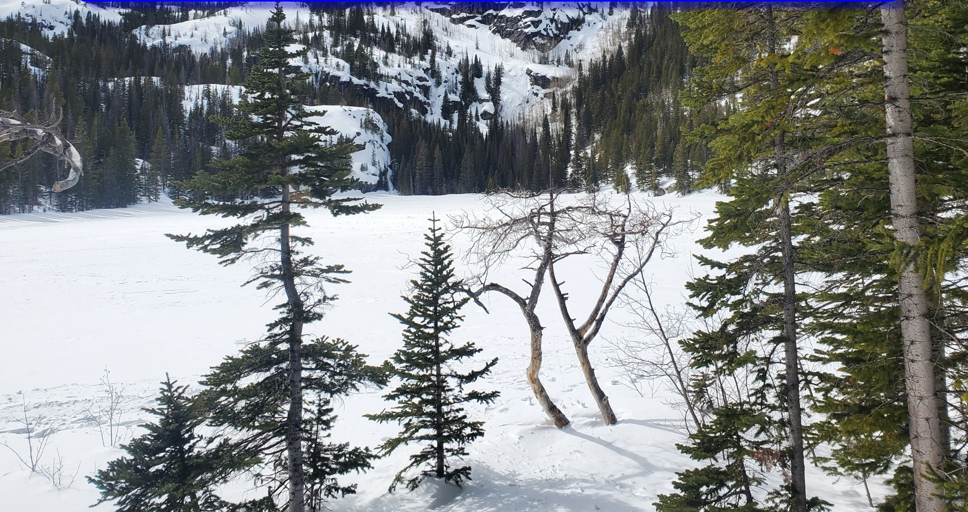
*Theorem (Landry, Lapidus and L., 2020)*

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((C(\mathscr{SG}_n), \mathscr{H}_n, \mathbb{D}_n), (C(\mathscr{SG}_\infty), \mathscr{H}_\infty, \mathbb{D}_\infty)) = 0.$$

- the approximating graphs converge in the *Gromov–Hausdorff distance* to $\mathscr{SG}_\infty$ *with its intrinsic the geodesic distance*;
- the functional calculi of the Dirac operators converge;
- the entire spectral geometry of the gasket is recovered as the limit of finite graph geometries.

# *Quantized Calculus on Fuzzy, Quantum Tori*

Let $\mathbb{Z}_k^d := \prod_{j=1}^d \mathbb{Z}/k_j\mathbb{Z}$ with $k := (k_1, \ldots, k_d) \in \mathbb{N} \cup \{\infty\}$ and $\mathbb{Z}/\infty\mathbb{Z} = \mathbb{Z}$. Let $\sigma$ be a 2-cocycle of $\mathbb{Z}_k^d$.



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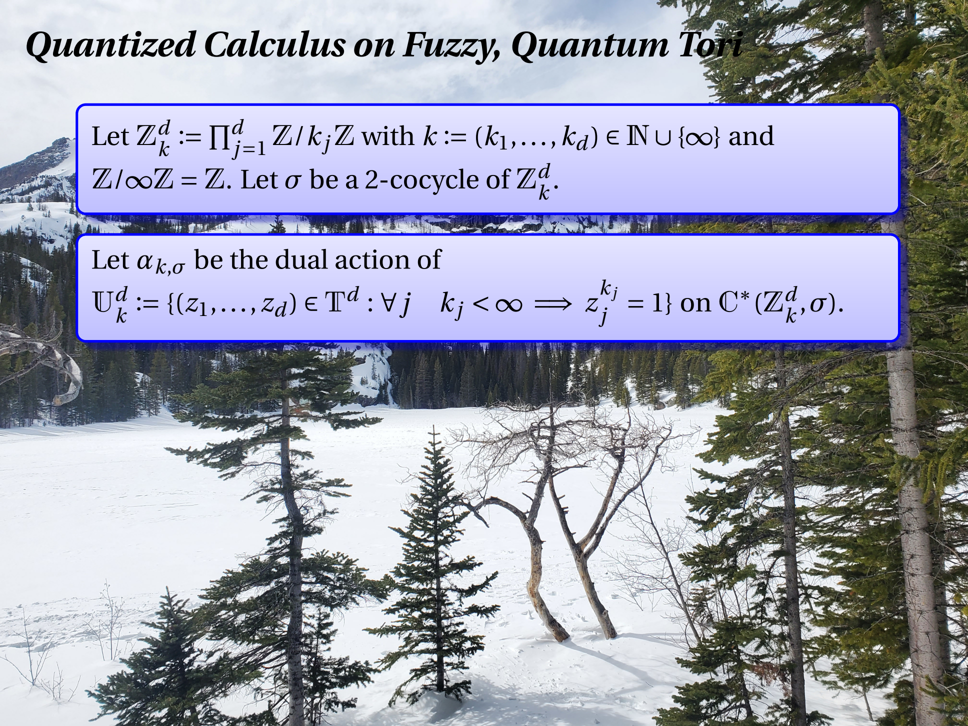
Let $\alpha_{k,\sigma}$ be the dual action of $\mathbb{U}_k^d := \{(z_1, \ldots, z_d) \in \mathbb{T}^d : \forall j \quad k_j < \infty \implies z_j^{k_j} = 1\}$ on $\mathbb{C}^*(\mathbb{Z}_k^d, \sigma)$.



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The natural calculus on $\mathbb{C}^*(\mathbb{Z}^d_k,\sigma)$ is encoded in a gradient defined for $a$ in a dense *-subalgebra as follows:

- $\partial^j_{k,\sigma} : a \mapsto \lim_{t\to 0} \frac{1}{t}\left(\alpha^{\exp(it)}_{k,\sigma}(a) - a\right)$ if $k_j = \infty$;

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- $\partial^j_{k,\sigma} : a \mapsto \frac{k_j}{2\pi}\left(\alpha^{\lambda_{j,k}}_{k,\sigma}(a) - a\right)$ when $k_j < \infty$ with $\lambda_j = (1, \ldots, 1, \exp(2i\pi k_j^{-1}), 1, \ldots, 1)$,

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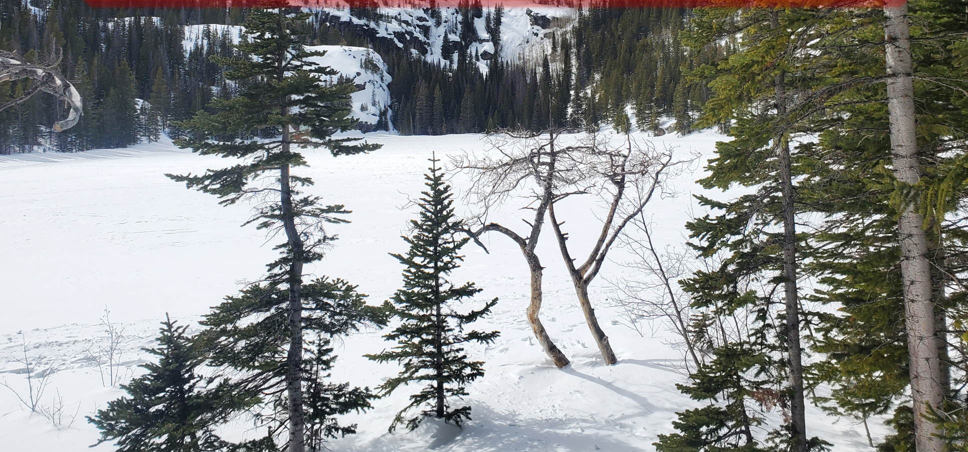
- $\mathrm{grad}_{k,\sigma}(a) := \sum_{j=1}^d \partial_{k,\sigma}^j(a) \otimes \gamma_j$.

# *Spectral Triples for Fuzzy Tori*

### *Leibniz Identity Fails*

The finite difference operators $\partial^j_{k,\sigma}$ are *not* derivations on $C^*(\mathbb{Z}^d_k,\sigma)$ when $k_j < \infty$, so the gradient *is not* representable using a spectral triple.



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*Three possible approaches*

1. *Change the discretized calculus* using commutators, as in math. phys. *Problem:* limit is not the usual flat spectral triple.
2. *Adjust our definition of a spectral triples **and** our discretized calculus:* truncations. *Problems:* operator systems instead of C*-algebras, *no more fuzzy tori*, unclear motivation for use of commutators.
3. *Adjust our definition of a spectral triple **in order to keep** our discretized calculus:* introduce a twist. *Problems:* none ☺ just need a minor adjustment to the spectral propinquity.

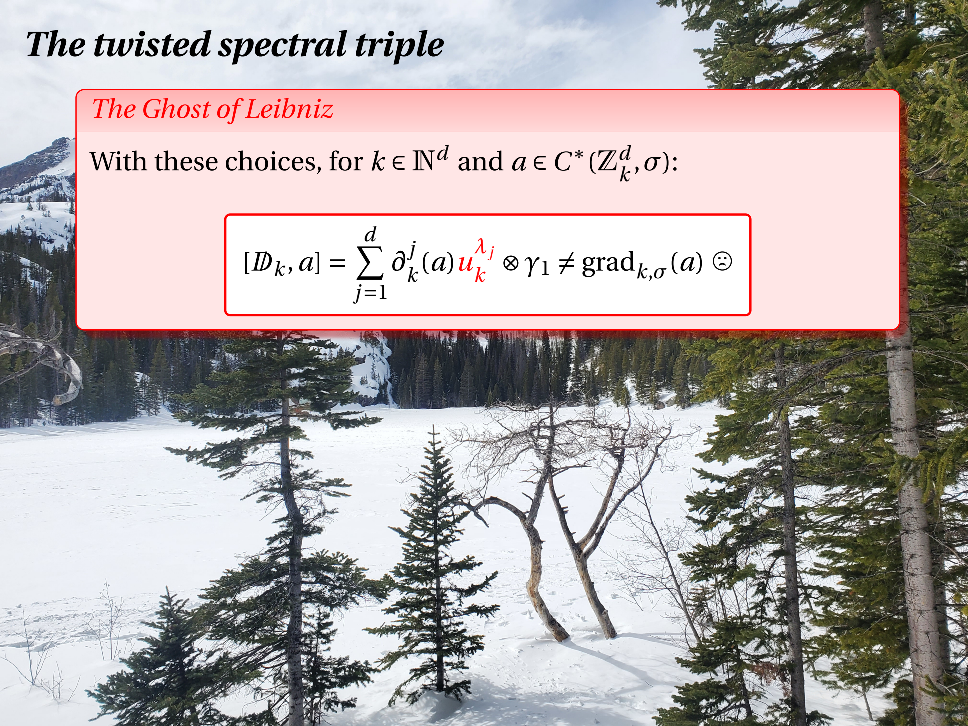
# *Flat (Quantum) Torus Approximations*

Start with the covariant representation on $\ell^2(\mathbb{Z}^d_k)$:

- for all $n \in \mathbb{Z}^d_k$, set $W^n_{k,\sigma}\xi(m) = \sigma(m,n)\xi(m-n)$, and $C^*(\mathbb{Z}^d_k,\sigma) = C^*(\{W^n_{k,\sigma} : n \in \mathbb{Z}^d_k\})$;
- for all $z \in \mathbb{U}^d_k$, set $u^z_k\xi(m) = z^m\xi(m)$ with $z^{\cdot}$ the character of $\mathbb{Z}^d_k$ given by $z$, so $u^z_k a u^{-z}_k = \alpha^z_{k,\sigma}(a)$ for all $a \in \mathbb{C}^*(\mathbb{Z}^d_k,\sigma)$.

The Hilbert space analogue of our calculus is generated by:

- $\partial^j_k\xi(m) := m_j\xi(m)$ if $k_j = \infty$ and $m = (m_1,\ldots,m_d)$,
- $\partial^j_k := \frac{k_j}{2\pi}\left(u^z_k - 1\right)$ if $k_j < \infty$,
- $\boxed{\mathbb{D}_k\xi := \sum_{j=1}^{d} \partial^j_k \otimes \gamma_j}$ on (a dense subspace of) $\ell^2(\mathbb{Z}^d_k,\mathbb{C}^d)$.



# *The twisted spectral triple*

*The Ghost of Leibniz*

With these choices, for $k \in \mathbb{N}^d$ and $a \in C^*(\mathbb{Z}^d_k, \sigma)$:

$$\boxed{[\not{D}_k, a] = \sum_{j=1}^{d} \partial_k^j(a)\, u_k^{\lambda_j} \otimes \gamma_1 \neq \mathrm{grad}_{k,\sigma}(a) \ ☹}$$

# The twisted spectral triple

## The Ghost of Leibniz

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## The Exorcism Twist

If for all $a \in C^*(\mathbb{Z}^d_k, \sigma)$, we set

$$\boxed{\rho_{k,\sigma}(a) := a + \sum_{k_j < \infty} \frac{2\pi}{k_j} \partial^j_{k,\sigma}(a)(\partial^j_k \otimes \gamma_j) \not{D}_k^{-1}} \in \mathfrak{B}(\ell^2(\mathbb{Z}^d_k, \mathbb{C}^d))$$

then $\rho$ is *linear* and

$$\boxed{[\not{D}_k, a]_{\rho_{k,\sigma}}} = \not{D}_k a - \rho_{k,\sigma}(a) \not{D}_k \boxed{= \mathrm{grad}_{k,\sigma}(a) \otimes 1 \smile}$$

# Twisted Spectral Triples

## Definition

A *twisted spectral triple* $(\mathfrak{A}, \mathscr{H}, \mathbb{D}, \rho)$ consists of a C*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$, a self-adjoint operator $\mathbb{D}$ defined on a dense subspace $\mathrm{dom}(\mathbb{D})$ of $\mathscr{H}$ with compact resolvent, and a *linear* map

$$\rho : \mathfrak{A}_{\mathbb{D}} \to \mathfrak{B}(\mathscr{H})$$

defined on a dense subspace $\mathfrak{A}_{\mathbb{D}}$ of $\mathfrak{A}$, such that if $a \in \mathfrak{A}_{\mathbb{D}}$ then $a\,\mathrm{dom}(\mathbb{D}) \subseteq \mathrm{dom}(\mathbb{D})$, and the twisted commutator

$$\boxed{[\mathbb{D}, a]_{\rho} := \mathbb{D}a - \rho(a)\mathbb{D}}$$

is bounded.

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is bounded. A *spectral triple* $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is a twisted spectral triple $(\mathfrak{A}, \mathscr{H}, \mathbb{D}, \mathrm{id}_{\mathfrak{A}})$ with twist the identity of $\mathfrak{A}$; in that case we would insist that $\mathfrak{A}_{\mathbb{D}}$ be a dense *-subalgebra.

# *Metric Twisted Spectral Triples*

*Definition (L. 26)*

A *$K$-metric twisted spectral triple* $(\mathfrak{A}, \mathscr{H}, \mathbb{D}, \rho)$, for $K \geqslant 1$, is a twisted spectral triple such that

1. if we define
$$\forall a \in \mathfrak{A}_{\mathbb{D}} \quad \mathsf{L}(a) := \left|\left|\left| [\mathbb{D}, a]_\rho \right|\right|\right|_{\mathscr{H}}$$
then the Monge-Kantorovich metric $\mathsf{mk}_{\mathsf{L}}$ metrizes the weak* topology on $\mathscr{S}(\mathfrak{A})$,
2. for all $a \in \mathfrak{A}_{\mathbb{D}}$,
$$\left\| \rho(a) \right\|_{\mathfrak{B}(\mathscr{H})} \leqslant K \max\{\mathsf{L}(a), \|a\|_{\mathfrak{A}}\},$$
3. $\rho$ has a closed graph for the product of the topology of the norm on $\mathfrak{A}$ with the weak operator topology on $\mathfrak{B}(\mathscr{H})$,
4. $\rho(1) = 1$.

# *The Spectral Propinquity with a twist*

*Definition (The Spectral Propinquity (L., 20, 22, **26**))*

The *spectral propinquity* $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1, \mathscr{H}_1, \mathbb{D}_1, \rho_1), (\mathfrak{A}_2, \mathscr{H}_2, \mathbb{D}_2, \rho_2))$ is the infimum of $\varepsilon > 0$ such that there exists a tunnel

$$(\mathscr{H}_1, \mathsf{D}_1, \mathfrak{A}_1, \mathsf{L}_{\mathbb{D}_1}, \mathbb{C}, 0) \xleftarrow{(\Psi_1, \pi_1, \theta_1)} (\mathscr{P}, \mathsf{T}, \mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \mathfrak{E}, \mathsf{L}_{\mathfrak{E}}) \xrightarrow{(\Psi_2, \pi_2, \theta_2)} (\mathscr{H}_1, \mathsf{D}_2, \mathfrak{A}_2, \mathsf{L}_{\mathbb{D}_2}, \mathbb{C}, 0)$$

such that $\chi(\pi_1) \vee \chi(\theta_1) \vee \chi(\pi_2) \vee \chi(\theta_2) \leqslant \varepsilon$, some appropriate measurement of how far the orbits $(\exp(it\mathbb{D}_1))_{t>0}$ and $(\exp(it\mathbb{D}_2))_{t>0}$ is less than $\varepsilon$,

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# *The Spectral Propinquity with a twist*

*Definition (The Spectral Propinquity (L., 20, 22, **26**))*

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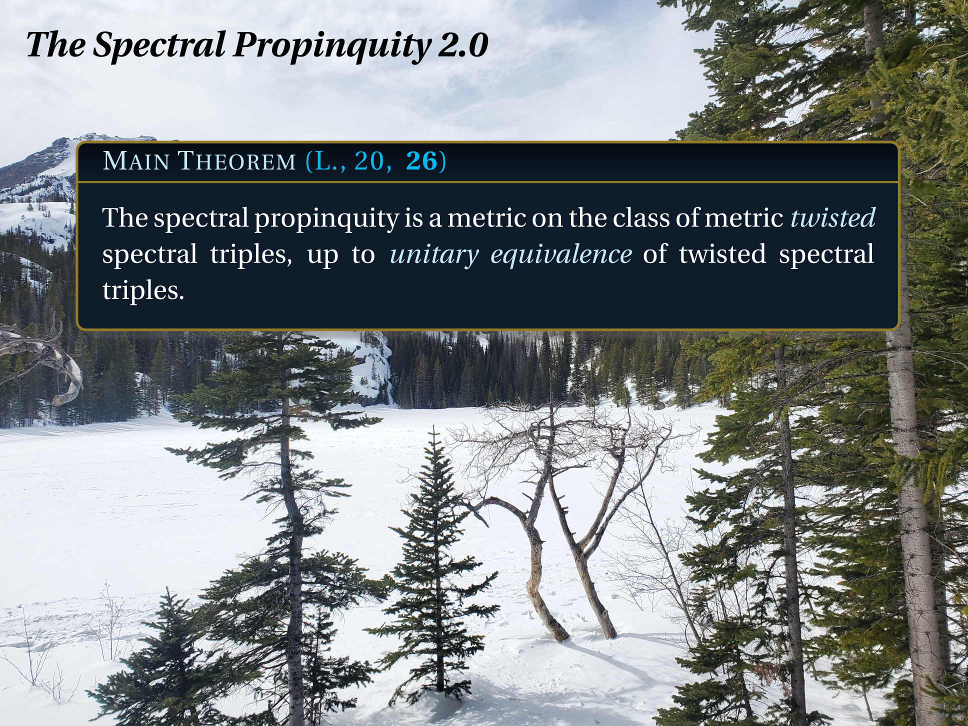
such that $\chi(\pi_1)\vee\chi(\theta_1)\vee\chi(\pi_2)\vee\chi(\theta_2)\leqslant\varepsilon$, some appropriate measurement of how far the orbits $(\exp(it\mathbb{D}_1))_{t>0}$ and $(\exp(it\mathbb{D}_2))_{t>0}$ is less than $\varepsilon$, and

$$\mathsf{Haus}_{\mathsf{mk}_{\mathsf{T}}}\Big(\Big\{\big(\langle\rho_1(\pi_1(d))\circ\Psi_1(\cdot),\omega\rangle_{\mathscr{P}}\big)_{\mathsf{L}_{\mathfrak{D}}(d)\leqslant 1} : \mathsf{T}(\omega)\leqslant 1\Big\}, \Big\{\big(\langle\rho_2(\pi_2(d))\circ\Psi_2(\cdot),\omega\rangle_{\mathscr{P}}\big)_{\mathsf{L}_{\mathfrak{D}}(d)\leqslant 1} : \mathsf{T}(\omega)\leqslant 1\Big\}\Big)\leqslant\varepsilon$$

where $\mathsf{mk}_{\mathsf{T},J}((\varphi_j)_{j\in J},(\psi)_{j\in J}) := \sup_{j\in J,\mathsf{T}(\omega)\leqslant 1}|\varphi_j(\omega)-\psi_j(\omega)|$.



# *The Spectral Propinquity 2.0*

**Main Theorem (L., 20, 26)**

The spectral propinquity is a metric on the class of metric *twisted* spectral triples, up to *unitary equivalence* of twisted spectral triples.

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Two twisted spectral triples $(\mathfrak{A}, \mathscr{H}, \mathbb{D}, \rho_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathscr{J}, \mathbb{V}, \rho_{\mathfrak{B}})$ are *unitary equivalent* when there exists a unitary $U : \mathscr{H} \rightarrow \mathscr{J}$ such that

- $U \operatorname{dom}(\mathbb{D}) = \operatorname{dom}(\mathbb{V})$,
- $U \mathbb{D} U^* = \mathbb{V}$,
- $a \in \mathfrak{A} \mapsto U a U^*$ is a $*$-isomorphism onto $\mathfrak{B}$,
- $\rho_{\mathfrak{B}}(U a U^*) = U \rho_{\mathfrak{A}}(a) U^*$.

# *The Spectral Propinquity 2.0*

**MAIN THEOREM (L., 20, 26)**

The spectral propinquity is a metric on the class of metric *twisted* spectral triples, up to *unitary equivalence* of twisted spectral triples.

The metric convergence, for the propinquity, and the spectral convergence of the Dirac operator *still hold unchanged.*

# *Application to fuzzy tori*

**MAIN THEOREM (L., 26)**

If $k_n \to (\infty,\ldots,\infty)$ and $\sigma_n \to \sigma$ pointwise on $\mathbb{Z}^d$, then

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}\Big( (C^*(\mathbb{Z}^d_{k_n},\sigma_n), \ell^2(\mathbb{Z}^d_k,\mathbb{C}^d), \not{D}_k, \rho_{\sigma_n,k_n}), \\ (C^*(\mathbb{Z}^d,\sigma), \ell^2(\mathbb{Z}^d,\mathbb{C}^d), \not{D}_\infty) \Big) = 0$$

where $(C^*(\mathbb{Z}^d,\sigma), \ell^2(\mathbb{Z}^d,\mathbb{C}^d), \not{D}_\infty)$ is the *usual* flat spectral triple.

# What if we insist on ordinary spectral triples?

*Theorem (L., 21)*

Let $k_n = (n, \ldots, n)$, $\sigma_n = \exp(i\pi k_n^{-1}\langle\cdot, J\cdot\rangle)$, and

$$\not{D}_n := \sum_{j=1}^{d} \frac{n}{\pi}\left(\left[W^{e_j}_{k_n,\sigma_n} + W^{e_j}_{k_n,\sigma_n}, \cdot\right] \otimes \gamma_{2j-1} + \left[W^{e_j}_{k_n,\sigma_n} - W^{e_j}_{k_n,\sigma_n}, \cdot\right] \otimes \gamma_{2j}\right)$$

where $C^*(\mathbb{Z}^d_{k_n}, \sigma_n)$ acts on $\ell^2(\mathbb{Z}^d_{k_n})$ on both the left and the right as $a \mapsto Ja^*J$ with $J\xi : m \mapsto \overline{\xi(-m)}$.
If we set on $\mathbb{T}^d = \{(\exp(it\theta_j))_{j=1}^d : \theta_1, \ldots, \theta_d \in [0, 2\pi)\}$ and

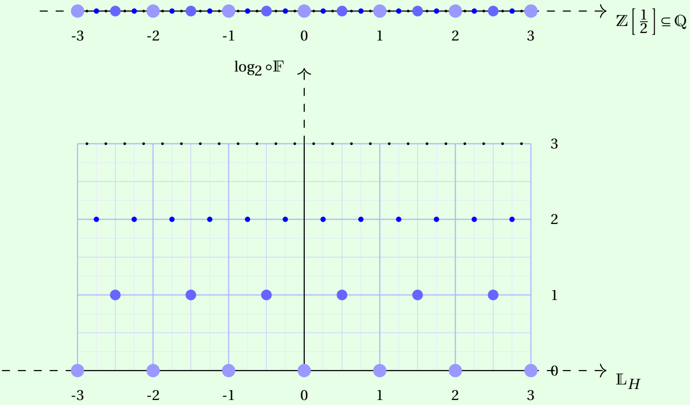
$$\not{D}_{\text{weird}} = \sum_{j=1}^{d} \cos(\theta_j)\partial_{\theta_j} \otimes \gamma_{2j-1} + \sin(\theta_j)\partial_{\theta_j} \otimes \gamma_{2j}, \text{ then}$$

$$\lim_{n\to\infty} \Lambda^{\text{spec}}((C^*(\mathbb{Z}^d_{k_n}, \sigma_n), \ell^2(\mathbb{Z}^d_{k_n}, \mathbb{C}^{2d}), \not{D}_n), (C(\mathbb{T}^d), \ell^2(\mathbb{T}^d, \mathbb{C}^{2d}), \not{D}_{\text{weird}})) = 0.$$

# Inductive Limits

## Example: Noncommutative Solenoids

Set $G_\infty := \mathbb{Z}\left[p^{-1}\right]^2$ and $G_n := \left(\frac{1}{p^n}\mathbb{Z}\right)^2$. We choose $\mathbb{L} := |\cdot|$ and $f : n \mapsto p^n$.



# *Inductive Limits II*

Let $G_\infty = \bigcup_{n\in\mathbb{N}} G_n$ be an Abelian discrete group where $G_n \uparrow G_\infty$ are subgroups, $\mathbb{L}$ a length on $G_\infty$ with $\mathsf{Haus}_{\mathbb{L}}(G_n, G_\infty) \xrightarrow{n\to\infty} 0$, and $\mathbb{F} : g \in G_\infty \mapsto f(\min\{n \in \mathbb{N} : g \in G_n\})$ with $f : \mathbb{N} \uparrow [0,\infty)$. Let

$$\slashed{D} := M_{\mathbb{L}} \otimes \gamma_1 + M_{\mathbb{F}} \otimes \gamma_2$$

on $\left\{\xi \in \ell^2(G_\infty, E) : \sum_{g\in G_\infty} (\mathbb{L}(g)^2 + \mathbb{F}(g)^2) \left\|\xi(g)\right\|_E^2 < \infty\right\}$. Let $\sigma$ be a 2-cocycle on $G_\infty$.

*Theorem (Farsi, L., Packer, 24)*

Assume $G_\infty$ is Abelian, and that $\mathbb{L}+\mathbb{F}$ is a length function such that

$$\exists C > 0\ \forall r \geqslant 1 \quad |G_\infty[2r]| \leqslant C \cdot |G_\infty[r]|.$$

Then $(C^*(G_n,\sigma), \ell^2(G_n) \otimes E, \slashed{D}_n)$ is metric for all $n \in \mathbb{N}$, and

$$(C^*(G_n,\sigma), \ell^2(G_n) \otimes E, \slashed{D}_{\ell^2(G_n)}) \xrightarrow[\Lambda^{\mathrm{spec}}]{n\to\infty} (C^*(G_\infty,\sigma), \ell^2(G_\infty) \otimes E, \slashed{D}_\infty).$$

# *Inductive Limits III*

*Example: Bunce-Deddens Algebras*

Set $G_\infty := \mathbb{Z}(p^\infty) \times \mathbb{Z}$ where $\mathbb{Z}(p^\infty) := \left\{z \in \mathbb{C} : \exists n \in \mathbb{N} \quad z^{(p^n)} = 1\right\}$. Let $G_n := \mathbb{Z}/p^n \times \mathbb{Z}$. Let $\sigma((\zeta, z), (\eta, y)) = \eta^z$.

*Example: UHF*

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with bounded integer ratios. Set $G_\infty := \{z \in \mathbb{C} : \exists n \in \mathbb{N} \quad z^{x_n} = 1\}$. Let $G_n := \{z \in \mathbb{C} : z^{x_n} = 1\}$.