Finite Dimensional Approximations of Spectral Triples on Quantum tori

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GPS

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New C-algebraic constructions*

What are the properties of classes of quantum metric spaces, and in particular, what does metric geometry tells us about noncommutative spaces?

A geometry for the space of quantum spaces

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- Motivated by mathematical physics.
- The focus is on various generalizations of the *Gromov-Hausdorff* distance.

GPS

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\boxed{\mathsf{L}(f) = \sup\left\{\frac{|f(x)-f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y\right\}}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\mathsf{mk}_{\mathsf{L}}(\mu,\nu) = \sup\left\{ \left| \int_X f\, d\mu - \int_X f\, d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.$$

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,

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4. The *weak* topology on $\mathscr{S}(\mathfrak{A})$* is metrized by the *Monge-Kantorovich metric* mk_{L}, defined $\forall \varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

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6. L is lower semi-continuous wrt $\|\cdot\|_{\mathfrak{A}}$.

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We call L an *L-seminorm*.

Some examples

- *Rieffel, 98*: unital C*-algebras with ergodic metric compact group actions,
- *Ozawa, Rieffel, 02*: word-Hyperbolic discrete group C*-algebras with Connes' length function spectral triple,
- *Li, 03*: Connes-Landi-du Bois-Voilette's spheres wrt spectral triple,
- *Hawkins, Skalski, White, Zacharias, 13*: C*-crossed products for equicontinuous actions wrt spectral triple,
- *L., Aguilar*: AF algebras with faithful tracial state,
- *L., 16*: Curved quantum tori, with metric from spectral triple,
- *L., 16*: Perturbations (conformal and otherwise) of metric spectral triples,
- *Christ, Rieffel, 16*: Nilpotent group C*-algebras with Connes' length function spectral triple,
- *Aguilar, Kaad, 17*: Podles spheres,
- *Kaad, Kyed, 19*: C*-crossed products for actions of $\mathbb{Z}$ by "Lipschitz" morphisms (twisted spectral triple),
- …

Lipschitz morphisms

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $(\exists k)$ $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $(\exists k)$ $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Lipschitz morphisms are arrows for a category over quasi-Leibniz quantum compact metric spaces.

Quantum Isometries

If $X \subset (Z, \mathsf{m})$ and $f \in \mathfrak{sa}(C(X))$:

- if $g \in C(Z)$ and $g\restriction_X = f$, the $\mathrm{Lip}(f) \leqslant \mathrm{Lip}(g)$,
- there exists $g \in \mathfrak{sa}(C(Z))$ with $g\restriction_X = f$ and $\mathrm{Lip}(f) = \mathrm{Lip}(g)$ (McShane, 34).

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Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that:

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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The dual maps of quantum isometries are indeed isometries for the Monge-Kantorovich metric.

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \middle| \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Edwards, 75; Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"'] \arrow[dr, "\pi_{\mathfrak{B}}"] & \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{tikzcd}$$

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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Definition *(The extent of a tunnel, L. 13,14)*

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

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The *dual propinquity* $\Lambda_F^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.}$$

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Theorem (L., 13)

The *dual propinquity* Λ^*_F, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ by:

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is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

Examples of Convergence

Example

- *L., 13* proved that fuzzy tori converge to quantum tori and quantum tori form a continuous family, for a natural metric from the dual actions,
- *Rieffel, 15* has constructed full matrix approximations of $C(S^2)$ using metrics arising from actions of $SU(2)$.
- *L. and Packer, 16* proved continuity for the family of noncommutative solenoids,
- *Aguilar, L., 15* proved certain continuity results for AF algebras,
- *L., 15* proved continuity for curved quantum tori,
- *L., 15* proved continuity for conformal deformations,
- *L., 15* proved that nuclear quasi-diagonal quantum compact metric spaces are limits of finite dimensional quantum compact metric spaces,
- and more …

Standard description of a noncommutative geometry

Definition (Connes, 89)

A $\boxed{\textit{spectral triple}}$ $(\mathfrak{A}, \mathscr{H}, D\!\!\!/)$ is given by:

1. a unital C*-algebra $\mathfrak{A}$,
2. a Hilbert space $\mathscr{H}$ on which $\mathfrak{A}$ acts,
3. a self-adjoint operator $D\!\!\!/$ on $\mathrm{dom}(D\!\!\!/) \subseteq \mathscr{H}$, such that:
   - $D\!\!\!/ \pm i$ has a compact inverse,
   - $\mathfrak{A}^1 = \{a \in \mathfrak{A} : a \cdot \mathrm{dom}(D\!\!\!/) \subseteq \mathrm{dom}(D\!\!\!/), [D\!\!\!/, a]$ has bounded closure$\}$ is dense in $\mathfrak{A}$.

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3. a self-adjoint operator $\not{D}$ on $\mathrm{dom}(\not{D}) \subseteq \mathscr{H}$, such that:
   - $\not{D} \pm i$ has a compact inverse,
   - $\mathfrak{A}^1 = \{a \in \mathfrak{A} : a \cdot \mathrm{dom}(\not{D}) \subseteq \mathrm{dom}(\not{D}), [\not{D}, a]$ has bounded closure$\}$ is dense in $\mathfrak{A}$.

If $(\mathfrak{A}, \mathscr{H}, D)$ is a spectral triple and if for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, we set:

$$\boxed{\mathsf{mk}_D(\varphi,\psi) = \sup\left\{\left|\varphi(a) - \psi(a)\right| : a \in \mathfrak{sa}(\mathfrak{A}), |||[D,a]|||_{\mathscr{H}} \leqslant 1\right\}}$$

then mk_D is an extended pseudo-distance on $\mathscr{S}(\mathfrak{A})$.

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$$\mathsf{mk}_D(\varphi, \psi) = \sup\left\{ \left|\varphi(a) - \psi(a)\right| : a \in \mathfrak{sa}(\mathfrak{A}), |||[D, a]|||_{\mathscr{H}} \leqslant 1 \right\}$$

then mk_D is an extended pseudo-distance on $\mathscr{S}(\mathfrak{A})$.

A metric condition

We want that mk_D induces the weak* topology on $\mathscr{S}(\mathfrak{A})$.

Standard description of a noncommutative geometry

Definition (Connes, 89)

A $\boxed{\textit{spectral triple}}$ $(\mathfrak{A}, \mathscr{H}, \mathcal{D})$ is given by:

1. a unital C*-algebra $\mathfrak{A}$,
2. a Hilbert space $\mathscr{H}$ on which $\mathfrak{A}$ acts,
3. a self-adjoint operator $\mathcal{D}$ on $\mathrm{dom}(\mathcal{D}) \subseteq \mathscr{H}$, such that:
   - $\mathcal{D} \pm i$ has a compact inverse,
   - $\mathfrak{A}^1 = \{a \in \mathfrak{A} : a \cdot \mathrm{dom}(\mathcal{D}) \subseteq \mathrm{dom}(\mathcal{D}), [\mathcal{D}, a] \text{ has bounded closure}\}$ is dense in $\mathfrak{A}$.

Definition

A *metric spectral triple* $(\mathfrak{A}, \mathscr{H}, \mathcal{D})$ is a spectral triple such that, if

$$\forall a \in \mathfrak{sa}\left(\mathfrak{A}^1\right) \quad \boxed{\mathsf{L}_{\mathcal{D}}(a) = |||[\mathcal{D}, a]|||_{\mathscr{H}}}$$

then $(\mathfrak{A}, \mathsf{L}_{\mathcal{D}})$ is a *quantum compact metric space*.

Ingredients for Convergence of Spectral Triples

Let $(\mathfrak{A}, \mathscr{H}, \not{D})$ be a spectral triple.

A Metric

Connes proposed a noncommutative version of the *Monge-Kantorovich metric* on the state space $\mathscr{S}(\mathfrak{A})$:

$$\mathsf{mk}_{\not{D}} : \varphi, \psi \in \mathscr{S}(\mathfrak{A}) \mapsto \sup\left\{ \left|\varphi(a) - \psi(a)\right| : a \in \mathfrak{sa}(\mathfrak{A}), |||[\not{D}, a]|||_{\mathscr{H}} \leqslant 1 \right\}.$$

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A Sobolev Space

The space $\mathrm{dom}(\not{D})$ is a Banach space for the *graph norm*

$$\xi \in \mathrm{dom}(\not{D}) \mapsto \mathsf{D}(\xi) = \|\xi\|_{\mathscr{H}} + \|\not{D}\xi\|_{\mathscr{H}}.$$

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A Quantum Dynamics

Since $\not{D}$ is self-adjoint, it gives rise to an *action of* $\mathbb{R}$ *by unitaries* of $\mathscr{H}$:

$$t \in \mathbb{R} \longmapsto \exp(2i\pi t \not{D}).$$

D-norms

The Hilbert space $\mathscr{H}$ of a spectral triple $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is a $\mathfrak{A}$-module and a $\mathbb{C}$-Hilbert module.

Next question: modules

How far are modules over quantum compact metric spaces?

D-norms

The Hilbert space $\mathscr{H}$ of a spectral triple $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a $\mathfrak{A}$-module and a $\mathbb{C}$-Hilbert module.

Next question: modules

How far are modules over quantum compact metric spaces?

To this end we need a key ingredient: the graph norm of D

$$\boxed{\forall \xi \in \mathrm{dom}\,(\not{D}) \quad \mathsf{D}_{\not{D}}(\xi) = \|\xi\|_{\mathscr{H}} + \|\not{D}\xi\|_{\mathscr{H}}}$$

Theorem (L., 2018,9)

If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a metric spectral triple and $\mathsf{D}_{\not{D}}$ is the graph norm of $\not{D}$ then:

$$\{\xi \in \mathscr{H} : \mathsf{D}_{\not{D}}(\xi) \leqslant 1\} \text{ is compact in } \mathscr{H},$$

while $\mathsf{D}_{\not{D}}(a\xi) \leqslant (\mathsf{L}_{\not{D}}(a) + \|a\|_{\mathfrak{A}})\mathsf{D}_{\not{D}}(\xi)$ (with $\mathsf{L}_{\not{D}}(a) = |||[\not{D}, a]|||_{\mathscr{H}}$).

Metrical C-correspondences*

Definition (L. (16,18,19))

A *metrical C*-correspondence* $\Omega = (\mathscr{M}, \mathsf{D}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is given by:

1. two *quantum compact metric spaces* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$,
2. a *right C*-Hilbert module* $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ over $\mathfrak{B}$, which is also *a left module* over $\mathfrak{A}$, with the actions of $\mathfrak{A}$ and $\mathfrak{B}$ commuting,
3. D is a *norm* on a dense subspace of $\mathscr{M}$ such that:
    1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
    2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
    3. $\forall \eta, \omega \in \mathscr{M} \quad \max\{\mathsf{L}_{\mathfrak{B}}(\Re\langle\omega,\eta\rangle_{\mathscr{M}}), \mathsf{L}_{\mathfrak{B}}(\Im\langle\omega,\eta\rangle_{\mathscr{M}})\} \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$,
    4. $\forall \eta \in \mathscr{M} \quad \forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{D}(a\eta) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(b), \mathsf{D}(\eta))$.

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   3. $\forall \eta, \omega \in \mathscr{M} \quad \max\{\mathsf{L}_{\mathfrak{B}}(\Re\langle\omega,\eta\rangle_{\mathscr{M}}), \mathsf{L}_{\mathfrak{B}}(\Im\langle\omega,\eta\rangle_{\mathscr{M}})\} \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$,
   4. $\forall \eta \in \mathscr{M} \quad \forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{D}(a\eta) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(b), \mathsf{D}(\eta))$.

For our purpose, if $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a *metric spectral triple*, then

$$\mathrm{qvb}(\mathfrak{A}, \mathscr{H}, \not{D}) = (\mathscr{H}, \mathsf{D}_{\not{D}}, \mathbb{C}, 0, \mathfrak{A}, \mathsf{L}_{\not{D}})$$

is a *metrical C^*-correspondence*.

Metrical Tunnels

Let $(\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2)$ be wo metric spectral triples.

$$\begin{tikzcd}
 & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
(\mathfrak{A}_1, \mathsf{L}_1) & & (\mathfrak{A}, \mathsf{L}_2)
\end{tikzcd}$$

Figure: π_1,π_2 are quantum isometries

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Figure: π_1,π_2 are quantum isometries

$$\begin{tikzcd}
& & (\mathscr{J}, \mathsf{T}) \arrow[dll, "\theta_1"'] \arrow[d, dotted] \arrow[drr, "\theta_2"] & & \\
(\mathscr{H}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathbb{C} \oplus \mathbb{C}, \mathsf{Q}) \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & & (\mathscr{H}_2, \mathsf{D}_2) \arrow[d, dotted] \\
(\mathbb{C}, 0) & & & & (\mathbb{C}, 0)
\end{tikzcd}$$

Figure: $\pi_{\mathfrak{A}}$,$\pi_{\mathfrak{B}}$ are quantum isometries

We want (π_j, θ_j) to be a *module morphism* and $\mathsf{D}_j = \inf \mathsf{T} \circ \theta_j^{-1}$ for $j \in \{1, 2\}$.

Approximating metrical quantum vector bundles

$$
\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dddll, "\Pi_1"'] \arrow[dddrr, "\Pi_2"] \arrow[d, dotted] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[ddl, "\theta_1"'] \arrow[ddr, "\theta_2"] & & \\
 & & (\mathbb{C} \oplus \mathbb{C}, \mathsf{Q}) \arrow[u, dashed] \arrow[ddl, "\pi_1"] \arrow[ddr, "\pi_2"'] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{H}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{H}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathbb{C}, 0) & & (\mathbb{C}, 0) &
\end{tikzcd}
$$

Figure: A metrical modular tunnel between $(\mathfrak{A}_1, \mathscr{H}_1, \mathbb{D}_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, \mathbb{D}_2)$.

The modular propinquity

Definition (L.,16,18)

A *tunnel* τ from $\mathbb{A}_1 = (\mathscr{M}_1, \mathsf{D}_1, \mathfrak{A}_1, \mathsf{L}^1_{\mathfrak{A}}, \mathfrak{B}_1, \mathsf{L}^1_{\mathfrak{B}})$ to $\mathbb{A}_2 = (\mathscr{M}_2, \mathsf{D}_2, \mathfrak{A}_2, \mathsf{L}^2_{\mathfrak{A}}, \mathfrak{B}_2, \mathsf{L}^2_{\mathfrak{B}})$ is given by:

- a metrical C*-correspondence $(\mathscr{P}, \mathsf{D}, \mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \mathfrak{E}, \mathsf{L}_{\mathfrak{E}})$,
- a module morphism (θ_j, Θ_j) from $\mathscr{P}$ to $\mathscr{M}_j$ which preserves the inner product and such that the quotient of D via Θ_j is D_j, while θ_j is a quantum isometry,
- a quantum isometry $\pi_j : (\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}) \to (\mathfrak{B}_j, \mathsf{L}^j_{\mathfrak{B}})$ such that (π_j, Θ_j) is also a module morphism.

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The extent of τ is the maximum of the extents of $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2)$ and $(\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}, \pi_1, \pi_2)$.

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The extent of τ is the maximum of the extents of $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2)$ and $(\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}, \pi_1, \pi_2)$. The *propinquity* between $\mathbb{A}_1$ and $\mathbb{A}_2$ is the infimum of the extent of all possible tunnels between them.

The modular propinquity

Definition (L.,16,18)

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Theorem (L., 16,18)

The modular propinquity is a complete metric on metrical C*-correspondence, up to full quantum isometry.

Application to metric spectral triples

Theorem (L., 19)

For any metric spectral triple $(\mathfrak{A}, \mathscr{H}, \not{D})$, we set:

$$\mathrm{qvb}\,(A, \mathscr{H}, \not{D}) = (\mathscr{H}, \mathsf{D}, \mathbb{C}, 0, \mathfrak{A}, \mathsf{L})$$

where

- $\mathsf{L}(a) = |||[\not{D}, a]|||_{\mathscr{H}}$,
- $\mathsf{D}(\xi) = \|\xi\|_{\mathscr{H}} + \|\not{D}\xi\|_{\mathscr{H}}$.

The function $\Lambda^{\mathsf{mod}}(\mathrm{qvb}\,(\cdot), \mathrm{qvb}\,(\cdot))$ is a pseudo-metric on metric spectral triples such that:

$$\Lambda^{\mathsf{mod}}(\mathrm{qvb}\,(\mathfrak{A}, \mathscr{H}_{\mathfrak{A}}, \not{D}_{\mathfrak{A}}), \mathrm{qvb}\,(\mathfrak{B}, \mathscr{H}_{\mathfrak{B}}, \not{D}_{\mathfrak{B}})) = 0$$

if and only if there exists a full quantum isometry $\pi : \mathfrak{A} \to \mathfrak{B}$ and a unitary $U : \mathscr{H}_{\mathfrak{A}} \to \mathscr{H}_{\mathfrak{B}}$ such that:

$$U \not{D}_{\mathfrak{A}}^2 U^* = D_{\mathfrak{B}}^2 \text{ and } \pi(a) = U a U^*.$$

Group actions

The phase of a Dirac operator

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If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a spectral triple then:

$$\boxed{t \in \mathbb{R} \mapsto U_t = \exp(i t \not{D})}$$

is a strongly continuous action of $\mathbb{R}$ on $\mathscr{H}$ such that:

$$\forall t \in \mathbb{R}, \xi \in \mathscr{H} \quad \mathsf{D}(U_t \xi) = \|U_t \xi\|_{\mathscr{H}} + \|\not{D} U_t \xi\|_{\mathscr{H}} = \mathsf{D}(\xi).$$

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Next step

How do we incorporate convergence of group actions?

A proper monoid Gromov-Hausdorff distance

Let (G_1, δ_1) and (G_2, δ_2) be two proper metric monoids. A *ε-isometric isomorphism* $(\varsigma_1, \varsigma_2)$ from (G_1, δ_1) to (G_2, δ_2) is a pair of functions $\varsigma_1 : G_1 \to G_2$ and $\varsigma_2 : G_2 \to G_1$ such that for all $\{j, k\} = \{1, 2\}$:

$$\forall g, g' \in G_j \left[\frac{1}{\varepsilon}\right], h \in G_k \left[\frac{1}{\varepsilon}\right]$$

$$\boxed{\left|\delta_k(\varsigma_j(g)\varsigma_j(g'), h) - \delta_j(gg', \varsigma_k(h))\right| \leqslant \varepsilon,}$$

and ς_1, ς_2 are unital.

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$$\forall g, g' \in G_j \left[\frac{1}{\varepsilon}\right], h \in G_k \left[\frac{1}{\varepsilon}\right]$$

$$\boxed{\left|\delta_k(\varsigma_j(g)\varsigma_j(g'), h) - \delta_j(gg', \varsigma_k(h))\right| \leqslant \varepsilon,}$$

and ς_1, ς_2 are unital.
We then set:

$$\Upsilon((G_1, \delta_1), (G_2, \delta_2)) = \boxed{\inf\left\{\varepsilon > 0 | \exists (\varsigma, \kappa)\ \varepsilon\text{-iso-iso}\right\}.}$$

Υ is a *metric up to isometric isomorphism of proper monoid*, and it dominates the pointed Gromov-Hausdorff distance.

The Covariant Reach

Let (τ, τ') be a *metrical tunnel* from $(\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1)$ to $(\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2)$.
Write $\tau = (\mathscr{P}, \mathsf{TN}, (\Pi_1, \pi_2), (\Pi_2, \pi_2))$.

Let $\varepsilon > 0$. Let $(\varsigma_1, \varsigma_2)$ be an *iso-iso* from $\mathbb{R}$ to $\mathbb{R}$.

Definition (L., 18)

The *ε-covariant reach* $\rho(\tau|\varepsilon)$ of $(\tau, \tau', \varsigma_1, \varsigma_2)$ is

$$\boxed{\max_{\{j,k\}=\{1,2\}} \sup_{\substack{\xi \in \mathscr{H}_j \\ \mathsf{D}_j(\xi) \leqslant 1}} \inf_{\substack{\eta \in \mathscr{H}_k \\ \mathsf{D}_k(\eta) \leqslant 1}} \sup_{|t| \leqslant \frac{1}{\varepsilon}} \mathsf{k}_{\mathsf{TN}}\left(e^{it\not{D}_j}\xi, e^{i\varsigma_j(t)\not{D}_k}\eta\right),}$$

where

$$\boxed{\mathsf{k}_{\mathsf{TN}}(\xi, \eta) = \sup_{\substack{\omega \in \mathscr{P} \\ \mathsf{TN}(\omega) \leqslant 1}} \left| \langle \xi, \Pi_1(\omega) \rangle_{\mathscr{H}_1} - \langle \eta, \Pi_2(\omega) \rangle_{\mathscr{H}_2} \right|.}$$

The Spectral Propinquity

Definition (L., 18)

The *spectral propinquity* $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1), (\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2))$ between two spectral triples is

$$\max\left\{\frac{\sqrt{2}}{2}, \inf\left\{\varepsilon > 0 : \exists \tau \in \Upsilon \quad \max\{\chi(\tau), \rho(\tau|\varepsilon)\} \leqslant \varepsilon\right\}\right\}$$

where Υ is the set of all covariant tunnels between the given spectral triples (for the actions $t \in \mathbb{R} \mapsto e^{it\not{D}_j}$).

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Theorem (L., 18)

The following assertions are equivalent:

1. $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1,\mathscr{H}_1,\not{D}_1),(\mathfrak{A}_2,\mathscr{H}_2,\not{D}_2)) = 0$,
2. there exists a unitary $U : \mathscr{H}_1 \to \mathscr{H}_2$, and a *-isomorphism $\theta : \mathfrak{A}_1 \to \mathfrak{A}_2$, such that

$$U\not{D}_1 U^* = \not{D}_2 \text{ and } \theta = \mathrm{Ad}(U).$$

GPS

Spectral Triple on Sierpiński gasket

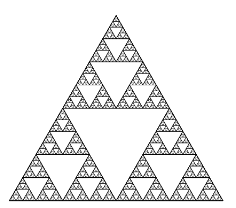


Figure: The Sierpiński gasket

The *Sierpiński gasket* is the attractor of the iterated function system $\{T_1, T_2, T_3\}$ with

$$T_j : x \in \mathbb{R}^2 \longmapsto \frac{1}{2}\left(x + v_j\right)$$

with

$$v_1 = (0,0), v_1 = (1,0), v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Spectral Triple on Sierpiński gasket

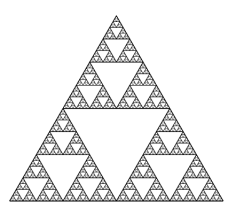
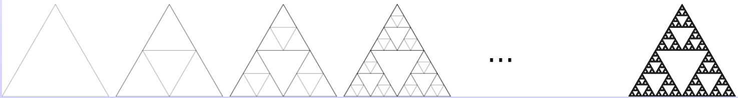


Figure: The Sierpiński gasket

The *Sierpiński gasket* can thus be seen as a limit, for the Hausdorff distance, of finite graphs, starting form the triangle $v_1 v_2 v_3$:



Spectral Triple on Sierpiński gasket

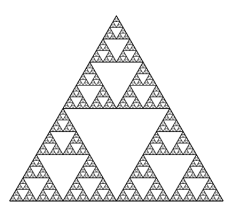


Figure: The Sierpiński gasket

Specifically, $\mathscr{S}\mathscr{G}_\infty$ is the limit, in the Hausdorff distance in $\mathbb{R}^2$, of the sequence of graphs:

1. We let $\mathscr{S}\mathscr{G}_0$ be the triangle $v_1 v_2 v_3$.
2. Let $\mathscr{S}\mathscr{G}_{n+1} = T_1\mathscr{S}\mathscr{G}_n \cup T_2\mathscr{S}\mathscr{G}_n \cup T_3\mathscr{S}\mathscr{G}_3$ for all $n \in \mathbb{N}$.

Spectral Triple on Sierpiński gasket

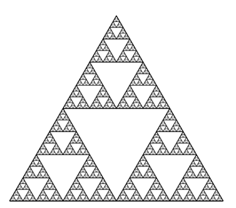


Figure: The Sierpiński gasket

Lapidus et al. constructed a spectral triple over the Sierpiński gasket. First, we define a spectral triple over $\mathfrak{CP} = \{f : [-1,1] \mapsto \mathbb{C} : f(-1) = f(1)\}$ by setting $\mathscr{J} = L^2(\mathfrak{CP}, \lambda)$, on which $\mathfrak{CP}$ acts by left multiplication, and ∂ is the closure of the linear map such that:

$$\boxed{\forall k \in \mathbb{Z} \quad \partial \exp(i\pi k \cdot) = i\pi \left(k + \frac{1}{2}\right) \exp(i\pi k \cdot).}$$

Spectral Triple on Sierpiński gasket

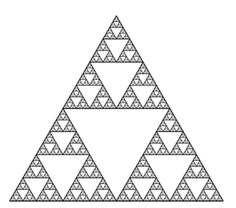


Figure: The Sierpiński gasket

We then define a spectral triple over $C([0,1])$ by identifying $C([0,1])$ with a C*-subalgebra of $\mathfrak{CP}$, sending $f \in C([0,1])$ to $x \in [-1,1] \mapsto f(|x|)$.

Spectral Triple on Sierpiński gasket

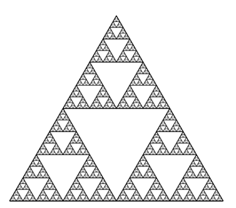


Figure: The Sierpiński gasket

Let $n \in \mathbb{N} \cup \{\infty\}$ (and write $3^\infty = \infty$). Let $\mathscr{H}_n = \oplus_{k=0}^{n} \oplus_{j=1}^{3^k} \mathscr{J}$, and for $\xi = (\xi_{k,j})_{\substack{k=0,\ldots,n\\ j=1,\ldots,3^k}} \in \mathscr{H}_n$ with all components in $\mathrm{dom}(\partial)$:

$$\boxed{\not{D}_n \xi = \left(2^k \partial \xi_{k,j}\right)_{\substack{k=0,\ldots,n\\ j=1,\ldots,3^k}} .}$$

$C(\mathscr{SG}_\infty)$ acts on $\mathscr{H}_n$ by multiplication by restrictions over edges in $\mathscr{SG}_n$.

Spectral Triple on Sierpiński gasket

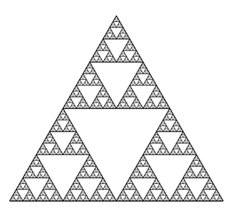
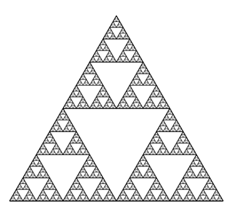


Figure: The Sierpiński gasket

Theorem (Lapidus et al., 08)

For all $n \in \mathbb{N} \cup \{\infty\}$, the triple $(C(\mathscr{S}\mathscr{G}_\infty), \mathscr{H}_n, \mathbb{D}_n)$ is a metric spectral triple, whose metric restricts to the *geodesic* distance on $\mathscr{S}\mathscr{G}_n$.

Spectral Triple on Sierpiński gasket



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Theorem (Landry, Lapidus, L., 20)

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((C(\mathscr{SG}_n), \mathscr{H}_n, \mathbb{D}_n), (C(\mathscr{SG}_\infty), \mathscr{H}_\infty, \mathbb{D}_\infty)) = 0.$$

Spectral Triples for Fuzzy Tori (Informal)

One example of our work deals with the clock and shift matrices:

$$\boxed{C_n = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & r_n & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & r_n^{n-1} \end{pmatrix} \text{ and } S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},}$$

with $r_n = \exp\left(\frac{2i\pi}{n}\right)$.

This pair of matrices encodes a *dynamical system*.

Note that $\boxed{S_n C_n S_n^* = r_n C_n}$ and $C^*(C_n) = C(\mathrm{Sp}(C_n))$. Thus, $C^*(C_n, S_n)$ is the C*-crossed product for the action of $\mathbb{Z}_n$ on $\mathrm{Sp}(C_n)$ by rotation of angle $\exp\left(\frac{2i\pi}{n}\right)$.

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with $r_n = \exp\left(\frac{2i\pi}{n}\right)$.

$C_n = \mathscr{F} S_n \mathscr{F}^*$ where $\mathscr{F}$ is the *discrete Fourier transform*, so if C_n is a position observable, then S_n it the momentum (up to scaling).

Thus $n[C_n, \cdot]$ and $n[S_n, \cdot]$, acting on $C^*(C_n, S_n)$, are discrete analogues of a moving frame of $\mathbb{T}^2$.

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with $r_n = \exp\left(\frac{2i\pi}{n}\right)$.

Let $\mathscr{H}_n$ be the Hilbert space given by $M_n(\mathbb{C})$ with inner product

$$\forall \xi, \eta \in M_n(\mathbb{C}) \quad \langle \xi, \eta \rangle = \frac{1}{n} \mathrm{Tr}(\xi^* \eta).$$

Of course, $C^*(C_n, S_n)$ acts on $\mathscr{H}_n$ both the left, via multiplication, and on the right, via multiplication by the adjoint.

Spectral Triples for Fuzzy Tori (Informal)

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$$C_n = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & r_n & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & r_n^{n-1} \end{pmatrix} \text{ and } S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

with $r_n = \exp\left(\frac{2i\pi}{n}\right)$.

We will study the spectral triple $(C^*(C_n, S_n), \mathscr{H} \otimes \mathbb{C}^4, \not{D}_n)$, where

$$\not{D}_n = \frac{n}{2}\Big([C_n + C_n^*, \cdot] \otimes \gamma_1 + i[C_n - C_n^*, \cdot] \otimes \gamma_2 + [S_n + S_n^*, \cdot] \otimes \gamma_3 + i[S_n^* - S_n, \cdot] \otimes \gamma_4\Big).$$

and $\gamma_j\gamma_s + \gamma_s\gamma_j = -2 1_{s=j}$ with $\gamma_1, \ldots, \gamma_4$ are 4×4 matrices.

Case Study: Spectral Triples on Quantum Tori (Informal)

Our work proposes, in particular, to prove that the *spectral triples on $C^*(C_n, S_n)$ converge to some spectral triple on $C(\mathbb{T}^2)$. Write $C(\mathbb{T}^2) = C^*(U, V)$ with $VU = UV$*, and U,V unitaries.

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$C^*(U,V)$ acts on $\ell^2(\mathbb{Z}^2)$ by

$$\forall \xi \in \ell^2(\mathbb{Z}^2) \quad U\xi : m \in \mathbb{Z}^2 \mapsto \xi(m-(1,0)) \text{ and } V\xi : m \mapsto \xi(m-(0,1)).$$

The natural Sobolev derivatives are given ($j \in \{1,2\}$) by

$$\partial_j \xi : (m_1, m_2) \mapsto i m_j \partial_j,$$

wherever this gives an element of $\ell^2(\mathbb{Z}^2)$.

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The heuristics is that and element of the form $\sum_{j,k\in F} a_{j,k} C_n^j S_n^k$ in $C^*(C_n, S_n)$ is an approximation of an element of the form $\sum_{j,k\in F} a_{j,k} U^j V^k$, and then:

$$n\left[S_n, \sum a_{j,k} C_n^j S_n^k\right] = \frac{S_n\left(\sum a_{j,k} C_n^j S_n^k\right) S_n^* - \sum a_{j,k} C_n^j S_n^k}{\frac{1}{n}} S_n$$
$$\rightsquigarrow \partial_1\left(\sum a_{j,k} U^j V^k\right) V.$$

Similarly, $n[C_n, \cdot]$ is presumably analogue to $\partial_2(\cdot) U$.

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We will work with the Dirac-type operator:

$$\boxed{\not{D} = \Re(U)\partial_2 \otimes \gamma_1 + \Im(U)\partial_2 \otimes \gamma_2 + \Re(V)\partial_1 \otimes \gamma_3 + \Im(V)\partial_1 \otimes \gamma_4.}$$

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We use $\Re(a) = \frac{a+a^*}{2}$, $\Im(a) = \frac{a-a^*}{2i}$, and 4×4 matrices $\gamma_1, \ldots, \gamma_4$ such that $\gamma_j\gamma_k + \gamma_k\gamma_j = 1_{j=k}$. This operator is densely defined, self-adjoint, with compact resolvent, over a dense subspace of $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$.

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For all $f \in C(\mathbb{T}^2)$ such that $f \cdot \mathrm{dom}(\not{D}) \subseteq \mathrm{dom}(\not{D})$ and $[\not{D}, f \otimes 1_{\mathbb{C}^2}]$ is bounded, we set

$$\boxed{\mathsf{L}(f) = \left|\left|\left| [\not{D}, f \otimes 1_{\mathbb{C}^4}] \right|\right|\right|_{\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^4}.}$$

$\boxed{\left(C(\mathbb{T}^2), \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^4, \not{D}\right)}$ is a spectral triple, and $\boxed{\left(C(\mathbb{T}^2), \mathsf{L}\right)}$ is a quantum compact metric space.

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Our work proposes, in particular, to prove that the *spectral triples on $C^*(C_n, S_n)$ converge to some spectral triple on $C(\mathbb{T}^2)$. Write $C(\mathbb{T}^2) = C^*(U, V)$ with $VU = UV$*, and U, V unitaries.

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Theorem (L., 21)

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}\left(\left(C^*(C_n, S_n), \mathscr{H}_n \otimes \mathbb{C}^4, \not{D}_n\right), \left(C(\mathbb{T}^2), \ell^2(\mathbb{Z}^{2d}, \mathbb{C}^4), \not{D}\right)\right) = 0.$$

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
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- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Indiana Univ. Math. J.* **66** (2017) 5, 1707–1753, Arxiv: 1501.06121
- *The modular Gromov-Hausdorff propinquity*, F. Latrémolière, Dissertationes Math. 544 (2019), 70 pp. 46L89 (46L30 58B34)
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- *The Gromov-Hausdorff Propinquity for Metric Spectral Triples*, F. Latrémolière, submitted (2019), arXiv:1811.10843.
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