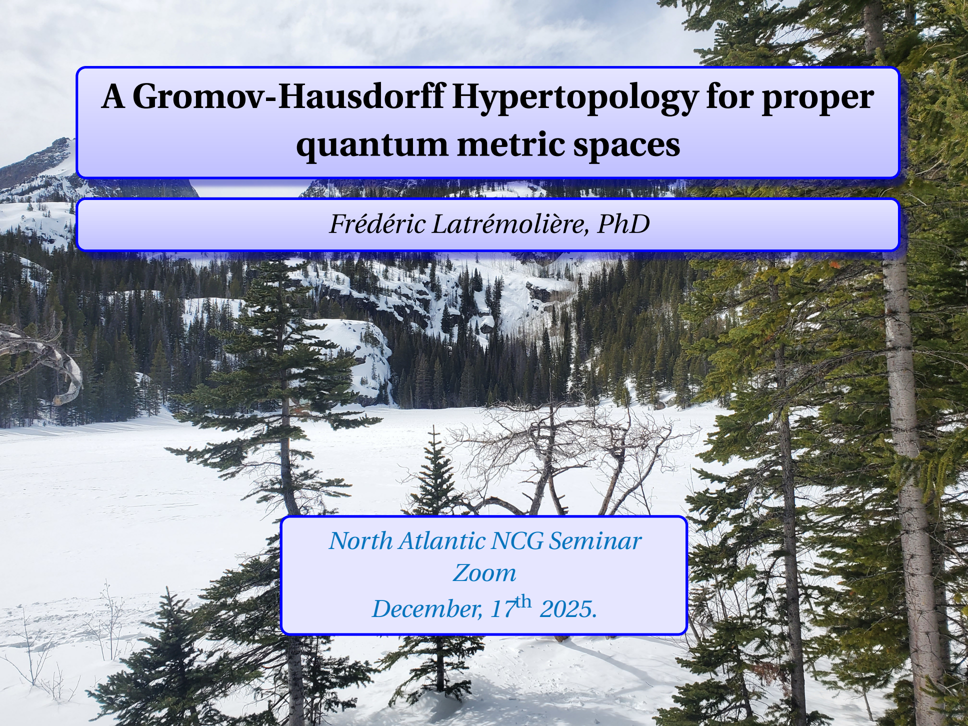

A Gromov-Hausdorff Hypertopology for proper quantum metric spaces

Frédéric Latrémolière, PhD

North Atlantic NCG Seminar
Zoom
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Motivation and Context

The Foundational Allegory

Noncommutative geometry is founded on the allegory that certain *noncommutative algebras* are analogues of *algebras of functions over some type of spaces*.

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P. Dirac, The principles of quantum mechanics, p. 26

We shall see later than, in spite of this fundamental difference [noncommutativity], the dynamical variables of quantum mechanics still have many properties in common with their classical counterparts and it will be possible to build up a theory of them closely analogous to the classical theory and forming a beautiful generalization of it.

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A. Connes, Compact Metric Spaces, Fredholm modules, and Hyperfiniteness, ETDS, 89

In this paper we shall work at the C^*-algebra level, i.e. we are dealing with noncommutative analogues of compact spaces, such as the duals of discrete groups. We shall develop for such spaces the analogue of the notion of *metric* on a compact space, examples include orginary compact Riemannian manifolds as well as the duals of discrete groups Γ on which a word length function is specified.

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M. Rieffel, Metrics on states from actions of compact groups, Doc. Math. 98

Connes has shown us that Riemannian metrics on non-commutative spaces (C^*-algebras) can be specified by generalized Dirac operators. Although in this setting there is no underlying manifold on which one then obtains an ordinary metric, Connes has shown that one does obtain in a simple way an ordinary metric on the state space of a C^*-algebra, generalizing the Monge-Kantorovich metric on probability measures [...].

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Noncommutative **metric** *geometry* is founded on the allegory that certain *noncommutative algebras* are analogues of *algebras of* **Lipschitz** *functions over metric spaces*.

M. Rieffel, Gromov-Hausdorff distance for quantum metric spaces, MAMS, 01

When one looks at the theoretical physics literature which deals with string theory and related topics, one finds various statements to the effect that some sequence of operator algebras converges to another operator algebra. [. . . . All of this] suggests that there are metric considerations involved in their convergence of algebras, and that one is perhaps dealing with some kind of convergence for corresponding "quantum" metric spaces. Within the mathematical literature, the only widely used notion of convergence of orginary metric spaces of which I am aware is that given by the Gromov-Hausdorff distance between metric spaces.

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- Rieffel constructed a first analogue of the Gromov-Hausdorff distance for quantum compact metric spaces (00),
- *but* distance zero did not imply *-isomorphism, betraying some possible “structure loss”,
- Variants were devised to address this: Kerr’s distance using operator systems (01), Wei Wu using operator spaces (05), Li with various versions (03), Kerr and Li, …

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The propinquity

- (L. 13,14,15) introduced the *propinquity* on the class of quantum compact metric spaces constructed over unital C*-algebras with a (quasi-)Leibniz L-seminorm.
- The propinquity is a complete metric up to full quantum isometry (including *-isomorphism)
- It is topologically equivalent to the Gromov-Hausdorff distance on classical compact metric spaces.
- There is an analogue for the propinquity of Gromov's compactness theorem (L. 15).

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Examples

- matricial approximations of quantum tori (L. 13)
- matricial approximations of C(coadjoint orbits of SU_2) (Rieffel, 15),
- continuity of the quantum tori in their cocycle (L. 13),
- continuity of certain families of AF algebras (UHF, Effros-Shen) (Aguilar, L. 15),
- approximations of noncommutative solenoids (L., Packer 16),
- continuity of curved quantum tori (L. 16),

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The propinquity is the starting point for metrics on more involved structures, including *spectral triples*, leading to results about:

- Matrix models converging to quantum tori (L. 22),
- Approximations of spectral triples on certain fractals (Landry, Lapidus, L. 22),
- Collapse in differential geometry (Farsi, L. 23),
- Inductive limits of spectral triples (Farsi, Packer, L., 23),
- Semicontinuity of groups of isometries (Bassi, Farsi, Conti, , L., 23)
- Continuity of the spectra and more (L. 22).

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What about locally compact spaces?

The theory for *quantum locally compact metric spaces* is much more *elusive*. The entire world of convergence for quantum locally compact metric spaces is yet to be explored. This omission is really the consequence of the challenging realm of quantum locally compact metric spaces, where the clean picture of the compact situation completely blurs away.

Motivation and Context

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Our Goal

We wish to define a notion of *pointed proper quantum metric space* suitable for a generalization of the *Gromov-Hausdorff convergence*.

ArXiv

arXiv:2512.03573

Section 1

Proper Quantum Metric Spaces

The Classical Realm

If (X,d) be a *locally compact metric space*. For any $f : X \to \mathbb{C}$, we set

$$\boxed{\mathsf{L}_d(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x,y \in X, x \neq y\right\}}$$

allowing ∞.

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For any two *states* $\varphi,\psi \in C_0(X)$, the *Fortet-Mourier distance* between φ,ψ is

$$\boxed{\mathsf{bl}_{\mathsf{L}_d,M}(\varphi,\psi) := \sup\left\{|\varphi(f)-\psi(f)| : \mathsf{L}_d(f)\leqslant 1, \|f\|_{C_0(X)} \leqslant M\right\}}$$

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and the *Monge-Kantorovich metric* between φ, ψ is

$$\boxed{\mathsf{mk}_{\mathsf{L}_d}(\varphi, \psi) := \sup\left\{|\varphi(f) - \psi(f)| : \mathsf{L}_d(f) \leqslant 1\right\}}$$

The Weak Topology question I*

Theorem (Fortet-Mourier, 1953)

If (X,d) is a locally compact separable metric space, then the topology induced by the Fortet-Mourier distance defined on $\mathscr{S}(C_0(X))$ by setting for all $\varphi,\psi \in \mathscr{S}(C_0(X))$:

$$\boxed{\mathsf{bl}_{\mathsf{L}_{d,M}}(\varphi,\psi) := \sup\left\{|\varphi(f)-\psi(f)| : \mathsf{L}_d(f) \leqslant 1, \|f\|_{C_0(X)} \leqslant M\right\}}$$

is the weak* topology. Moreover, if (X,d) is *proper* and $x,y \in X$, then

$$\mathsf{bl}_{\mathsf{L}_d}(\delta_x,\delta_y) = \min\{d(x,y),M\}.$$

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Remark

Thus the Fortet-Mourier is a good foundation for the theory of quantum locally compact metric space, see L. 05.

The Noncommutative Forter-Mourier metric

Theorem (L. 05)

If $\mathfrak{A}$ is a *separable C*-algebra* and L is a seminorm defined on a dense subalgebra of $\mathfrak{A}$, then the *Forter-Mourier distance*:

$$\mathsf{bl}_{\mathsf{L}}(\varphi,\psi) := \sup\left\{|\varphi(a)-\psi(a)| : a \in \mathrm{dom}(\mathsf{L}), \max\{\mathsf{L}(a), \|a\|_{\mathfrak{A}}\} \leqslant 1\right\}$$

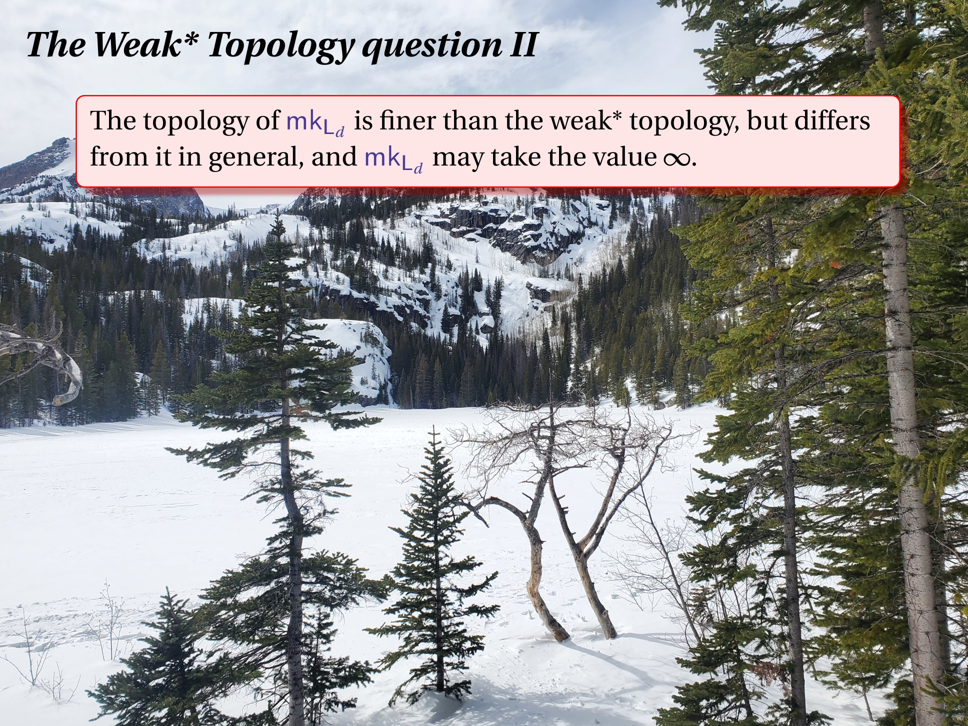
metrizes the *weak* topology* if, and only if, there *exists* $h \in \mathfrak{sa}(\mathfrak{A})$ with $h > 0$ such that

$$\boxed{\{hah : a \in \mathrm{dom}(\mathsf{L}), \max\{\mathsf{L}(a), \|a\|_{\mathfrak{A}}\} \leqslant 1\}}$$

is totally bounded (hence compact if closed since $\mathfrak{A}$ is complete).

The Weak Topology question II*

The topology of $\mathsf{mk}_{\mathsf{L}_d}$ is finer than the weak* topology, but differs from it in general, and $\mathsf{mk}_{\mathsf{L}_d}$ may take the value ∞.



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Theorem (Dobrushin, 70)

Let (X,d) be a separable locally compact metric space. If $S \subseteq \mathscr{S}(C_0(X))$, if $x \in X$, and if

$$\lim_{r\to\infty} \sup_{\mu\in S} \int_{y\in X:d(x,y)>r} d(x,y)\,d\mu(y) = 0,$$

then the restriction of $\mathsf{mk}_{\mathsf{L}_d}$ to S is the weak* topology.

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Corollary

If (X,d) is compact, then $\mathsf{mk}_{\mathsf{L}_d}$ induces the weak* topology on $\mathscr{S}(C(X))$, and moreover, for all $x,y \in X$, then

$$\mathsf{mk}_{\mathsf{L}_d}(\delta_x,\delta_y) = d(x,y).$$

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Theorem (L. 12, VERY Informal)

If $\mathfrak{A}$ is a separable C*-algebra and L is a norm modulo constants on a dense subspace $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$, then mk_{L} induces the weak* topology on "tame" if, and only if, there exists $h \in \mathfrak{sa}(\mathfrak{A})$, $h > 0$, and a state $\mu \in \mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$, such that

$$\left\{hah : a \in \mathfrak{u}\mathfrak{A}, \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$$

is totally bounded in $\mathfrak{A}$.

The Weak Topology question II*

Proposition

If (X,d) is a separable locally compact metric space, then there exists $\mu \in \mathscr{S}(C_0(X))$ such that

$$\left\{\varphi \in \mathscr{S}(C_0(X)) : \mathsf{mk}_{\mathsf{L}_d}(\varphi,\mu) < \infty\right\}$$

is weak* dense in $\mathscr{S}(C_0(X))$.

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The above property is implied by the condition that there exists $h \in C_0(X)$, $h > 0$ such that $\{hfh : f \in C_(X \cup \{\infty\}), \mathsf{L}_d(f) \leqslant 1, f(x_0) = 0\}$ is *bounded* for some (and hence any) $x_0 \in X$.

Locally Compact Quantum Metric Spaces 2.0

Definition (L. 05, 12, 13, 25)

$(\mathfrak{A}, \mathsf{L})$ is a *quantum locally compact metric space* when

1. L is a *hermitian norm modulo constants* defined on a dense *-subalgebra $\mathrm{dom}(\mathsf{L})$ of a *separable C*-algebra* $\mathfrak{A}$,

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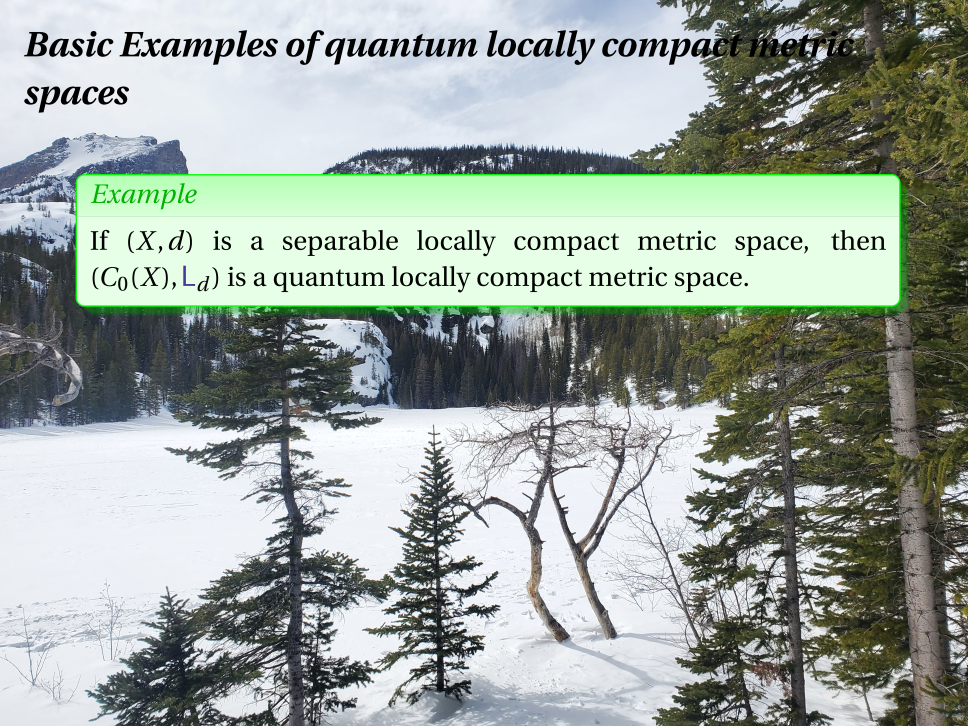
4. $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1\}$ is closed,
5. there exists $h \in \mathfrak{sa}(\mathfrak{A})$ strictly positive and a state $\mu \in \mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$ such that $\left\{hah \in \mathfrak{u}\mathfrak{A} : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$ is bounded.

Basic Examples of quantum locally compact metric spaces

Example

If (X, d) is a separable locally compact metric space, then $(C_0(X), \mathsf{L}_d)$ is a quantum locally compact metric space.



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Theorem (L. 25)

Let $(\mathfrak{A}, \mathsf{L})$ be a *quantum locally compact metric space*. The C*-algebra $\mathfrak{A}$ is *unital* if, and only if, $(\mathfrak{A}, \mathsf{L})$ is a (Leibniz) *quantum compact metric space*.

A metric on $c_0(\mathbb{Z}) \rtimes_{.+t} \mathbb{Z}$

Example

For all $f \in c_0(\mathbb{Z})$ and $(\xi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we define:

$$\boxed{f(\xi_n)_{n\in\mathbb{Z}} := (f(n)\xi_n)_{n\in\mathbb{Z}}.}$$

For all $(\xi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we define

$$\boxed{U(\xi_n)_{n\in\mathbb{Z}} \mapsto (\xi_{n-1})_{n\in\mathbb{Z}}.}$$

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If

$$\boxed{\mathsf{L} : f \in c_0(\mathbb{Z}) \mapsto \left|\!\left|\!\left| [U, f] \right|\!\right|\!\right|_{\ell^2(\mathbb{Z})}}$$

then $(c_0(\mathbb{Z}), \mathsf{L})$ is an quantum locally compact metric space, and $\mathbb{Z}$ is endowed with its usual metric.

$c_0(\mathbb{Z}) \rtimes_{\cdot+t} \mathbb{Z}$ *as a quantum locally compact metric space*

Example

The C*-algebra generated by $c_0(\mathbb{Z})$ and U^t is $c_0(\mathbb{Z}) \rtimes_{\cdot+t} \mathbb{Z}$.

$c_0(\mathbb{Z}) \rtimes_{\cdot + t} \mathbb{Z}$ *as a quantum locally compact metric space*

Example

The C*-algebra generated by $c_0(\mathbb{Z})$ and U^t is $c_0(\mathbb{Z}) \rtimes_{\cdot + t} \mathbb{Z}$. For all $(\xi_n)_{n\in\mathbb{Z}}$ with $(n\xi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we set:

$$\boxed{\frac{d(\xi_n)_{n\in\mathbb{Z}}}{dz} := (n\xi_n)_{n\in\mathbb{Z}}.}$$

We then define on a dense subspace of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$:

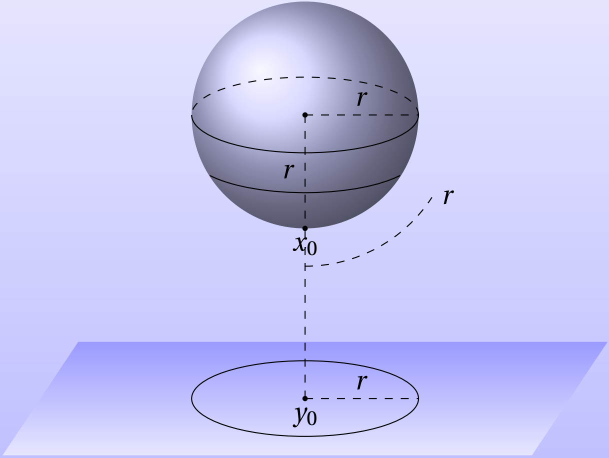
$$\boxed{\not{D} := [U, \cdot] \otimes \gamma_1 + \frac{d}{dz} \otimes \gamma_2.}$$

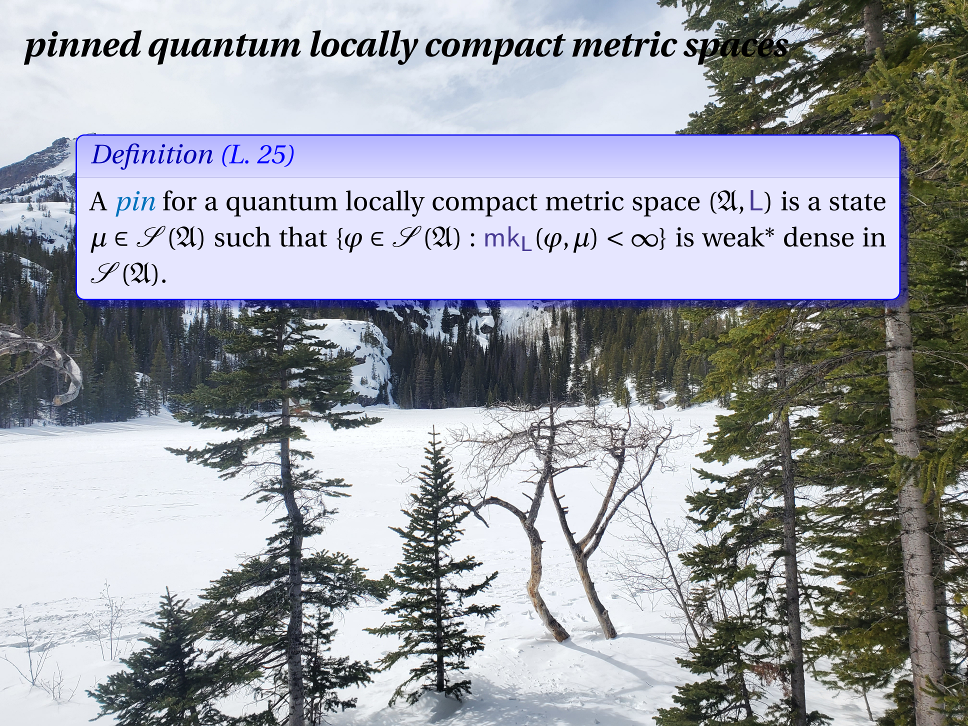
If

$$\boxed{\mathsf{L} : a \in c_0(\mathbb{Z}) \rtimes \mathbb{Z} \mapsto \left|\!\left|\!\left| [\not{D}, a \otimes 1_{\mathbb{C}^2}] \right|\!\right|\!\right|}$$

then $(c_0(\mathbb{Z}) \rtimes \mathbb{Z}, \mathsf{L})$ is an quantum locally compact metric space.

The Gromov-Hausdorff Distance





pinned quantum locally compact metric spaces

Definition (L. 25)

A *pin* for a quantum locally compact metric space $(\mathfrak{A}, \mathsf{L})$ is a state $\mu \in \mathscr{S}(\mathfrak{A})$ such that $\{\varphi \in \mathscr{S}(\mathfrak{A}) : \mathsf{mk}_{\mathsf{L}}(\varphi, \mu) < \infty\}$ is weak* dense in $\mathscr{S}(\mathfrak{A})$.

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There always exist a pin for any quantum locally compact metric space.

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Lemma (L. 25)

There always exist a pin for any quantum locally compact metric space.

Definition (L. 25)

A *pinned* quantum locally compact metric space $(\mathfrak{A}, \mathsf{L}, \mu)$ is a quantum locally compact metric space $(\mathfrak{A}, \mathsf{L})$ together with a pin $\mu \in \mathscr{S}(\mathfrak{A})$.

Lipschitz Sequences

Definition (L. 25)

Let $(\mathfrak{A}, \mathsf{L}, \mu)$ be a pinned quantum locally compact metric space. A sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(\mathsf{L})$ is *μ-Lipschitz-exhaustive* when:

1. $\lim_{n\to\infty} \|e_n\|_{\mathfrak{A}} = 1$,
2. $\lim_{n\to\infty} \mathsf{L}(e_n) = 0$,
3. $\lim_{n\to\infty} \mu(e_n) = 1$.

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If $(\mathfrak{A}, \mathsf{L})$ is a quantum locally compact metric space, and if $(e_n)_{n\in\mathbb{N}}$ is *Lipscthiz exhaustive*, then $(e_n)_{n\in\mathbb{N}}$ is an *approximate unit of $\mathfrak{A}$*.

Proof Sketch

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- $|\mu(e_n) - \varphi(e_n)| \leqslant \mathsf{mk}_{\mathsf{L}}(\varphi,\psi)\mathsf{L}(e_n) \xrightarrow{n\to\infty} 0$ for φ in a weak* dense subset S of $\mathscr{S}(\mathfrak{A})$,

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- Since $\lim_{n\to\infty}\mu(e_n) = 1$, we conclude $\lim_{n\to\infty}\varphi(e_n) = 1$ for $\varphi \in S$.
- Let $\varphi \in \mathscr{S}(\mathfrak{A})$. For $\varepsilon > 0$, there exists $\psi \in S$ with $\mathsf{bl}_{\mathsf{L}}(\varphi,\psi) < \frac{\varepsilon}{2}$ as bl_{L} *metrizes the weak* topology*, and $\exists N \in \mathbb{N} \quad n \geqslant N \implies |\psi(e_n) - 1| < \frac{\varepsilon}{2}$, so $|\varphi(e_n) - 1| < \varepsilon$. So $\lim_{n\to\infty}\varphi(e_n) = 1$.

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- So $\lim_{n\to\infty}\widehat{e_n}(\varphi) = 1$, where $\widehat{e_n} : \varphi \in \mathscr{S}(\mathfrak{A}) \mapsto \varphi(e_n)$. Similarly $(\widehat{e_n^2})_{n\in\mathbb{N}}$ converges pointwise to 1.

Proof Sketch

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If $(\mathfrak{A}, \mathsf{L})$ is a quantum locally compact metric space, and if $(e_n)_{n\in\mathbb{N}}$ is *Lipscthiz exhaustive*, then $(e_n)_{n\in\mathbb{N}}$ is an *approximate unit of* $\mathfrak{A}$.

- $|\widehat{e_n}(\varphi) - \widehat{e_n}(\psi)| \mathsf{L}(e_n) \mathsf{bl}_{\mathsf{L}}(\varphi, \psi)$, so $\{\widehat{e_n} : n \in \mathbb{N}\}$ is equicontinuous for bl_{L},

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- if $K \subseteq \mathscr{S}(\mathfrak{A})$ is weak* compact, as bl_{L} *metrizes the weak* topology*, we conclude by Arzéla-Ascoli that $(\widehat{e_n})_{n\in\mathbb{N}}$ and $(\widehat{e_n^2})_{n\in\mathbb{N}}$ converges uniformly to 1 on K,

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- If $h > 0$ in $\mathfrak{sa}(\mathfrak{A})$ and $p_n := \xi_{[1-n^{-1},1]}(h)$ then $\|p_n a p_n\|_{\mathfrak{A}''} = \sup_{\varphi(p_n)=1} |\varphi(a)|$; $\{\varphi : \varphi(p_n) = 1\}$ is weak* compact, so $\lim_{m\to\infty} \|p_n e_m a e_m p_n\|_{\mathfrak{A}} = \|p_n a p_n\|_{\mathfrak{A}}$ using Cauchy-Schwarz. Then use $(p_n h)_{n\in\mathbb{N}}$ cv to h.

Proper Quantum Metric Spaces

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Theorem (L. 25)

If (X, d) is a locally compact metric space, then $(C_0(X), \mathsf{L}_d)$ is a quantum locally compact metric space. Moreover, (X, d) is proper if, and only if $(C_0(X), \mathsf{L}_d)$ is a proper quantum metric space.

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Theorem (L. 25)

If $(\mathfrak{A}, \mathsf{L})$ is a quantum locally compact metric space and if there exists an *approximate unit* $(e_n)_{n\in\mathbb{N}}$ bounded for $\|\cdot\|_{\mathsf{L}} := \max\{\mathsf{L}, \|\cdot\|_{\mathfrak{A}}\}$, then $(\mathscr{S}(\mathfrak{A}), \mathsf{bl}_{\mathsf{L}})$ is a *complete metric space*.

Corollary (L. 25)

If $(\mathfrak{A}, \mathsf{L})$ is a proper quantum metric space, then $(\mathscr{S}(\mathfrak{A}), \mathsf{bl}_{\mathsf{L},1})$ is complete.

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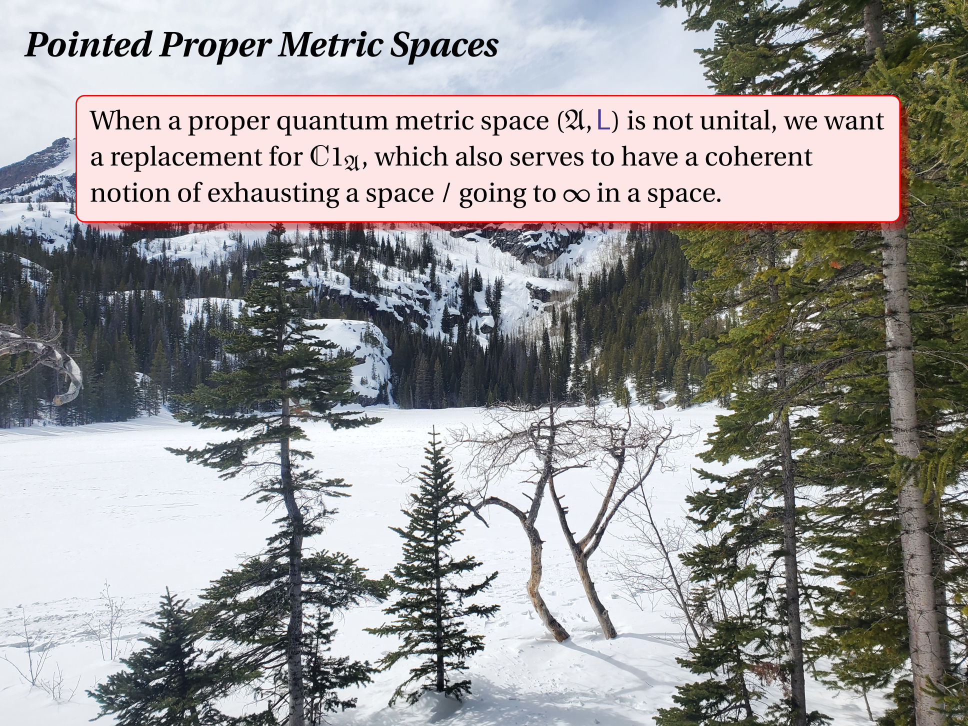
Theorem (L. 25)

If $(\mathfrak{A}, \mathsf{L})$ is a proper quantum metric space, then $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, or $\operatorname{diam}(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}) = \infty$ (and $\mathfrak{A}$ is not unital).

Pointed Proper Metric Spaces

When a proper quantum metric space $(\mathfrak{A}, \mathsf{L})$ is not unital, we want a replacement for $\mathbb{C}1_{\mathfrak{A}}$, which also serves to have a coherent notion of exhausting a space / going to ∞ in a space.



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Definition (L., 12)

A *topography* $\mathfrak{M}$ of a proper quantum metric space $(\mathfrak{A}, \mathsf{L})$ is an *Abelian* C*-subalgebra of $\mathfrak{A}$ containing a Lipschitz exhaustive sequence.

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Topographies were central in the study of the behavior of the Monge-Kantorovich metric and the noncommutative generalization of Dobrushin's result (L. 12).

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Definition (L. 25)

A pointed proper quantum metric space $(\mathfrak{A}, \mathsf{L}, \mathfrak{M}, \mu)$ is given by a proper quantum metric space $(\mathfrak{A}, \mathsf{L})$ with a topgraphy $\mathfrak{M}$ and a pin $\mu \in \mathscr{S}(\mathfrak{A})$ which restricts to a *character* of $\mathfrak{M}$.

Quantum Isometries and Tunnels

Let $(\mathfrak{A}, \mathsf{L})$ be a proper quantum metric space. For any $M > 0$, let $\|a\|_{\mathsf{L},M} := \max\{\frac{1}{M}\|a\|_{\mathfrak{A}}, \mathsf{L}(a)\}$, and
$\mathsf{bl}_{\mathsf{L},M}(\varphi,\psi) := \sup\left\{|\varphi(a)-\psi(a)| : \|a\|_{\mathsf{L},M} \leqslant 1\right\}$.

Definition (L. 25)

A *topographic quantum M-isometry* $\pi : (\mathfrak{D}, \mathsf{L}, \mathfrak{M}) \twoheadrightarrow (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}})$ is a proper *-epimorphism from $\mathfrak{D}$ onto $\mathfrak{A}$ such that:

1. $\mathsf{L}_{\mathfrak{A}} \circ \pi \leqslant \mathsf{L}$ on $\mathrm{dom}(\mathsf{L})$ and for all $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L})$, $\|a\|_{\mathsf{L}_{\mathfrak{A}},M} := \inf\left\{\|d\|_{\mathsf{L},M} : d \in \mathrm{dom}(\mathsf{L}), \pi(d) = a\right\}$,

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Remark

If π is a quantum M-isometry, then π^* is an *isometry* from $(\mathscr{S}(\mathfrak{A}), \mathsf{bl}_{\mathsf{L},M})$ into $(\mathscr{S}(\mathfrak{D}), \mathsf{bl}_{\mathsf{L},M})$.

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2. $\pi(\mathfrak{M}) = \mathfrak{M}_{\mathfrak{A}}$, and $\forall \varepsilon > 0$, $\forall a \in \mathfrak{M}_{\mathfrak{A}} \cap \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_{\mathfrak{A}})$, $\exists d \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L})$ such that $\mathsf{L}(d) \leqslant \mathsf{L}_{\mathfrak{A}}(a) + \varepsilon$ and $\|d\|_{\mathfrak{D}} \leqslant \|a\|_{\mathfrak{A}} + \varepsilon$.

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Section 2

A new hypertopology

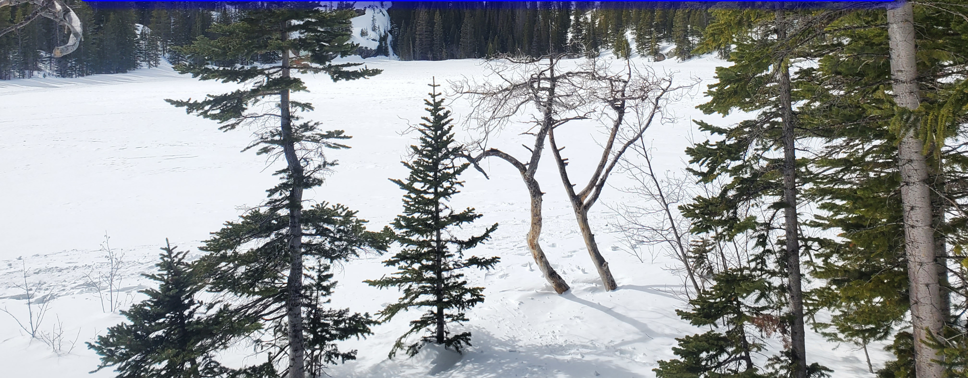
The Hausdorff distance

If (X, d) is a *metric space*, $\emptyset \neq A, B \subseteq X$, we define

$$\boxed{A \subseteq^d_\varepsilon B} \iff \forall x \in A \quad d(x, B) \leqslant \varepsilon$$

where

$$d(x, B) := \inf\{d(x, b) : b \in B\}.$$



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Definition (Hausdorff, 1893)

$$\boxed{\mathsf{Haus}_d(A,B) := \inf\left\{\varepsilon > 0 : A \subseteq_{\varepsilon}^{d} B \text{ and } B \subseteq_{\varepsilon}^{d} A\right\}.}$$

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$$\boxed{\mathsf{Haus}_{d}(A, B) := \inf\left\{\varepsilon > 0 : A \subseteq_{\varepsilon}^{d} B \text{ and } B \subseteq_{\varepsilon}^{d} A\right\}.}$$

Theorem (Hausdorff, 1893)

The *Hausdorff metric* is a metric on the hyperspace of closed subsets of X. If (X, d) is complete (resp. compact), then so is Haus_{d}.

The Gromov-Hausdorff Distance

If (Z, d) is a metric space, if $\emptyset \neq A, B \subseteq Z$, and if $a \in A$, $b \in B$, then define:

$$\delta_d((A,a),(B,b)) := \inf\left\{\varepsilon > 0 : A\left[a, \frac{1}{\varepsilon}\right] \subseteq^d_\varepsilon B \text{ and } B\left[b, \frac{1}{\varepsilon}\right] \subseteq^d_\varepsilon A\right\},$$

where $A[a,r] := \{x \in A : d(a,x) \leqslant r\}$.

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Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two *proper, pointed metric spaces* (X,x_0) and (Y,y_0) is:

$$\boxed{\begin{aligned}\mathsf{GH}((X,x_0),(Y,y_0)) := \inf\Big\{&\delta_d((j(X),j(x_0)),(r(Y),r(y_0))) : \\ &j : X \to Z, r : Y \to Z \text{ isometries into proper } (Z,d)\Big\}.\end{aligned}}$$

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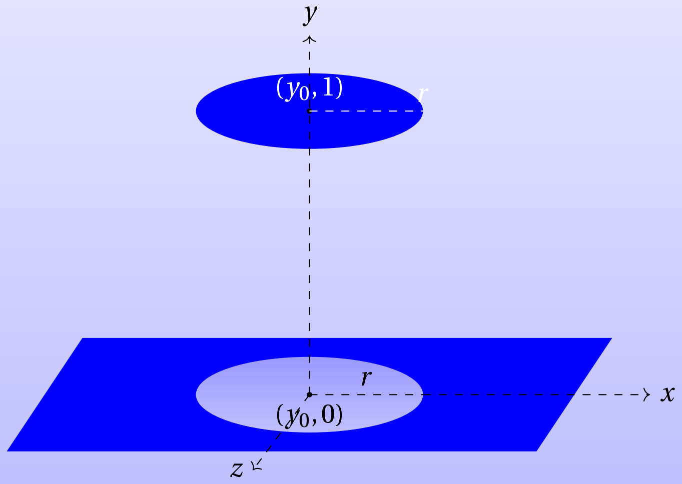
$$j : X \to Z, r : Y \to Z \text{ isometries into proper } (Z,d)\Big\}.$$

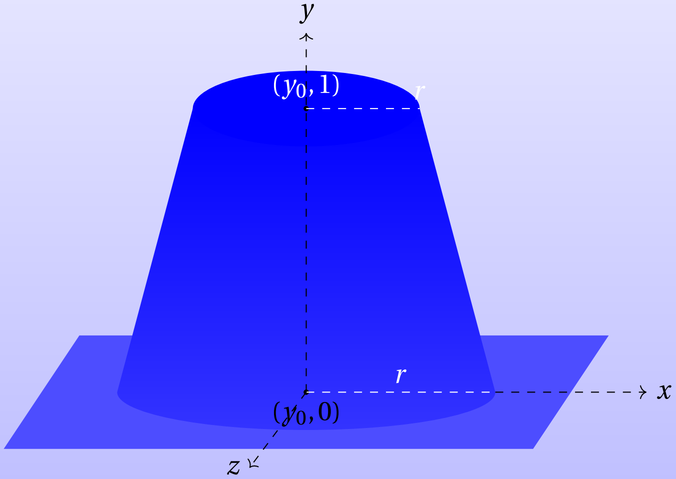
Theorem (Gromov, 81)

The Gromov-Hausdorff distance is a complete metric *up to full isometry of pointed proper metric spaces*.

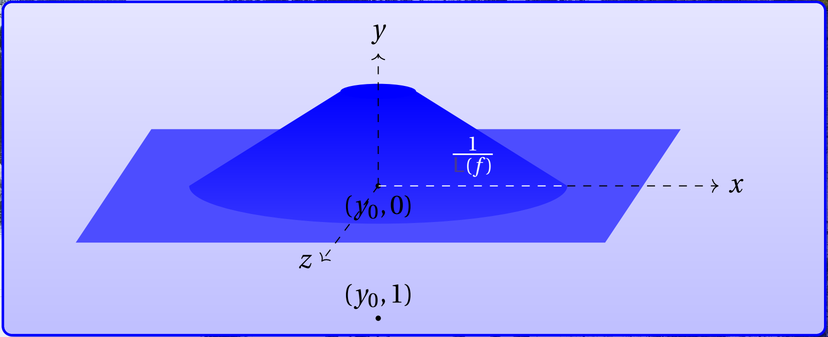
Localizing using projections



Localizing using Lipschitz functions



Localizing only with Lipschitz functions



Tunnels and Extent

Definition (L. 25)

An *M-tunnel* $\tau := (\mathfrak{D}, \mathsf{L}, \mathfrak{M}, \pi, \rho, e)$ is given by

1. a quantum locally compact metric space $(\mathfrak{D}, \mathsf{L}, \mathfrak{M})$,
2. two topographic quantum M-isometries
   $\pi : (\mathfrak{D}, \mathsf{L}, \mathfrak{M}) \to (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}})$ and $\rho : (\mathfrak{D}, \mathsf{L}, \mathfrak{M}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}})$,
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Definition (L. 25)

The *extent* of an M-tunnel $\tau := (\mathfrak{D}, \mathsf{L}, \mathfrak{M}, \pi, \rho, e)$ is given as the maximum of:

1. writing $e\varphi e := \varphi(e \cdot e)$,
   $\inf\left\{\varepsilon > 0 : e\mathscr{Q}(\mathfrak{D})e \subseteq_{\varepsilon}^{\mathsf{bl}_{\mathsf{L},M}} \pi^*(\mathscr{Q}(\mathfrak{A})), e\mathscr{Q}(\mathfrak{D})e \subseteq_{\varepsilon}^{\mathsf{bl}_{\mathsf{L},M}} \rho^*(\mathscr{Q}(\mathfrak{B}))\right\}$,

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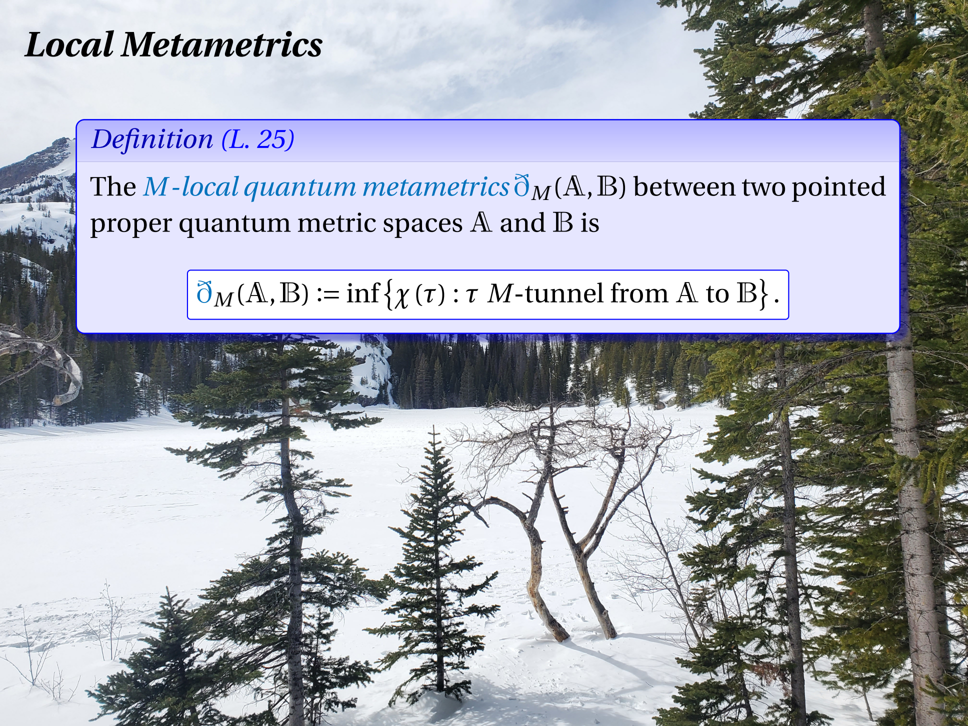
3. $\mathsf{mk}_{\mathsf{L}}(\mu_{\mathfrak{A}} \circ \pi, \mu_{\mathfrak{B}} \circ \rho)$.

Local Metametrics

Definition (L. 25)

The *M-local quantum metametrics* $\eth_M(\mathbb{A},\mathbb{B})$ between two pointed proper quantum metric spaces $\mathbb{A}$ and $\mathbb{B}$ is

$$\boxed{\eth_M(\mathbb{A},\mathbb{B}) := \inf\left\{\chi(\tau) : \tau \ M\text{-tunnel from } \mathbb{A} \text{ to } \mathbb{B}\right\}.}$$



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Theorem (L. 25)

For all pointed proper quantum metric spaces $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$,

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For all pointed proper quantum metric spaces $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$,

- $\eth_M(\mathbb{A},\mathbb{B}) = \eth_M(\mathbb{B},\mathbb{A})$,
- $\eth_M(\mathbb{A},\mathbb{B}) = 0$ if, and only if, there exists a full, pointed topograhic quantum M-isometry from $\mathbb{A}$ to $\mathbb{B}$.

Local Metametrics

Definition (L. 25)

The *M-local quantum metametrics* $\eth_M(\mathbb{A},\mathbb{B})$ between two pointed proper quantum metric spaces $\mathbb{A}$ and $\mathbb{B}$ is

$$\boxed{\eth_M(\mathbb{A},\mathbb{B}) := \inf\left\{\chi(\tau) : \tau\ M\text{-tunnel from } \mathbb{A} \text{ to } \mathbb{B}\right\}.}$$

Theorem (L. 25)

For all pointed proper quantum metric spaces $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$,

- $\eth_M(\mathbb{A},\mathbb{B}) = \eth_M(\mathbb{B},\mathbb{A})$,
- $\eth_M(\mathbb{A},\mathbb{B}) = 0$ if, and only if, there exists a full, pointed topograhic quantum M-isometry from $\mathbb{A}$ to $\mathbb{B}$.
- $\eth_M(\mathbb{A},\mathbb{B}) \leqslant$ $\left(1+\eth_M(\mathbb{D},\mathbb{B})\right)^2 \eth_M(\mathbb{A},\mathbb{D}) + \left(1+\eth_M(\mathbb{A},\mathbb{D})\right)^2 \eth_M(\mathbb{B},\mathbb{D})$.

A note on the triangle inequality

Let $\varepsilon > 0$,

$$\tau_1 : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}}) \xleftarrow{\pi_1} (\mathfrak{D}_1, \mathsf{L}_1, \mathfrak{M}_1, e_1) \xrightarrow{\rho_1} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$$

and

$$\tau_1 : (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}}) \xleftarrow{\pi_2} (\mathfrak{D}_2, \mathsf{L}_2, \mathfrak{M}_2, e_2) \xrightarrow{\rho_2} (\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}, \mathfrak{M}_{\mathfrak{E}}, \mu_{\mathfrak{E}}).$$

A note on the triangle inequality

Let $\varepsilon > 0$,

$$\tau_1 : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}}) \xleftarrow{\pi_1} (\mathfrak{D}_1, \mathsf{L}_1, \mathfrak{M}_1, e_1) \xrightarrow{\rho_1} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$$

and

$$\tau_1 : (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}}) \xleftarrow{\pi_2} (\mathfrak{D}_2, \mathsf{L}_2, \mathfrak{M}_2, e_2) \xrightarrow{\rho_2} (\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}, \mathfrak{M}_{\mathfrak{E}}, \mu_{\mathfrak{E}}).$$

We can make a new tunnel:

$$\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}}) \leftarrow (\mathfrak{D}_1 \oplus \mathfrak{D}_2, \mathsf{L}[T], \mathfrak{M}_1 \oplus \mathfrak{M}_2, f) \rightarrow (\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}, \mathfrak{M}_{\mathfrak{E}}, \mu_{\mathfrak{E}})$$

where in particular,

$$\mathsf{L}[T](d_1, d_2) := \max\left\{ \mathsf{L}_1(d_1), \mathsf{L}_2(d_2), \frac{1}{\varepsilon} \left\| \rho_1(d_1) - \pi_2(d_2) \right\|_{\mathfrak{B}} \right\}.$$

A note on the triangle inequality II

- Since π_2 is a topographic quantum isometry, we can find $e_2' \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_1)$ with $\mathsf{L}_1(e_2') \approx \mathsf{L}_2(e_2) \approx \chi(\tau_2)$ and $\left\|e_2'\right\|_{\mathfrak{D}_1} \approx \|e_2\|_{\mathfrak{D}_2}$.
- A consequence of our definitions is that $\|e_2\|_{\mathfrak{D}_2} \leqslant \sqrt{1+\chi(\tau_2)}$.
- So we can set $f_1 := e_1 e_2' \in \mathfrak{sa}(\mathfrak{M}_1)$. By the Leibniz property:

$$\mathsf{L}_1(e_1 e_2') \leqslant \sqrt{1+\chi(\tau_1)}\tau_2 + \sqrt{1+\chi(\tau_2)}\tau_1 + O(\varepsilon).$$

- If $\varphi \in \mathscr{Q}(\mathfrak{D}_1)$, then there exists $\psi \in \mathscr{Q}(\mathfrak{B})$ st $\mathsf{bl}_{\mathsf{L}_1}(\varphi \circ \pi_1(e_1 \cdot e_1), \psi \circ \rho_1) \leqslant \chi(\tau_1) + \varepsilon$. We can bound $\mathsf{bl}_{\mathsf{L}_1}(\varphi \circ \pi_1(e_1 e_2' \cdot e_2' e_1), \psi \circ \rho_1(e_2' \cdot e_2'))$ using Leibniz property, so extra factors appear.
- $\mathsf{bl}_{\mathsf{L}[T]}(\psi \circ \rho_1(e_2' \cdot e_2'), \psi \circ \pi_2(e_2 \cdot e_2)) \leqslant \varepsilon$ by construction,
- there exists θ such that $\mathsf{bl}_{\mathsf{L}_2}(\psi \circ \pi_2(e_2 \cdot e_2), \theta \circ \rho_2) \leqslant \chi(\tau_2) + \varepsilon$.

Target Sets and Coincidence property

Let $\tau := (\mathfrak{D}, \mathsf{L}, \mathfrak{M}, \pi, \rho, e)$ be an M-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$.

Definition (L. 13, 25)

For all $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_{\mathfrak{A}})$ and $l > \|a\|_{\mathsf{L},M}$, define:

$$\mathfrak{t}_\tau(a|l) := \left\{ \rho(d) : d \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}), \|d\|_{\mathsf{L},M} \leqslant l, \pi(d) = a \right\}.$$

Target Sets and Coincidence property

Let $\tau := (\mathfrak{D}, \mathsf{L}, \mathfrak{M}, \pi, \rho, e)$ be an M-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$.

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For all $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_{\mathfrak{A}})$ and $l > \|a\|_{\mathsf{L},M}$, define:

$$\mathfrak{t}_\tau(a|l) := \left\{\rho(d) : d \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}), \|d\|_{\mathsf{L},M} \leqslant l, \pi(d) = a\right\}.$$

Lemma (L. 25)

For all $a, a' \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_{\mathfrak{A}})$ and $l > \max\{\|a\|_{\mathsf{L}_{\mathfrak{A}},M}, \left\|a'\right\|_{\mathsf{L}_{\mathfrak{A}},M}\}$, if $b, b' \in \mathfrak{t}_\tau(a|l)$, then

$$\|ebe\|_{\mathfrak{B}} \leqslant \|a\|_{\mathfrak{A}} + l\chi(\tau),$$

and $b + tb' \in \mathfrak{t}_\tau\left(a + ta' \middle| (1+|t|)l\right)$, so

$$\mathsf{Haus}_{\mathfrak{B}}(e\mathfrak{t}_\tau(a|l)e, e\mathfrak{t}_\tau\left(a'\middle|l\right)e) \leqslant \left\|a - a'\right\|_{\mathfrak{B}} + 2l\chi(\tau).$$

Target Sets and Coincidence property

Let $\tau := (\mathfrak{D}, \mathsf{L}, \mathfrak{M}, \pi, \rho, e)$ be an M-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$.

Definition (L. 13, 25)

For all $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_{\mathfrak{A}})$ and $l > \|a\|_{\mathsf{L},M}$, define:

$$\mathfrak{t}_{\tau}(a|l) := \left\{\rho(d) : d \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}), \|d\|_{\mathsf{L},M} \leqslant l, \pi(d) = a\right\}.$$

Corollary (L. 25)

For all $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L})$, if $l > \|a\|_{\mathsf{L},M}$, then

$$\mathrm{diam}\left(e\mathfrak{t}_{\tau}(a|l)e, \mathfrak{B}\right) \leqslant 2l\chi(\tau).$$

Target Sets and Coincidence property

Let $\tau := (\mathfrak{D}, \mathsf{L}, \mathfrak{M}, \pi, \rho, e)$ be an M-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$.

Definition (L. 13, 25)

For all $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_{\mathfrak{A}})$ and $l > \|a\|_{\mathsf{L},M}$, define:

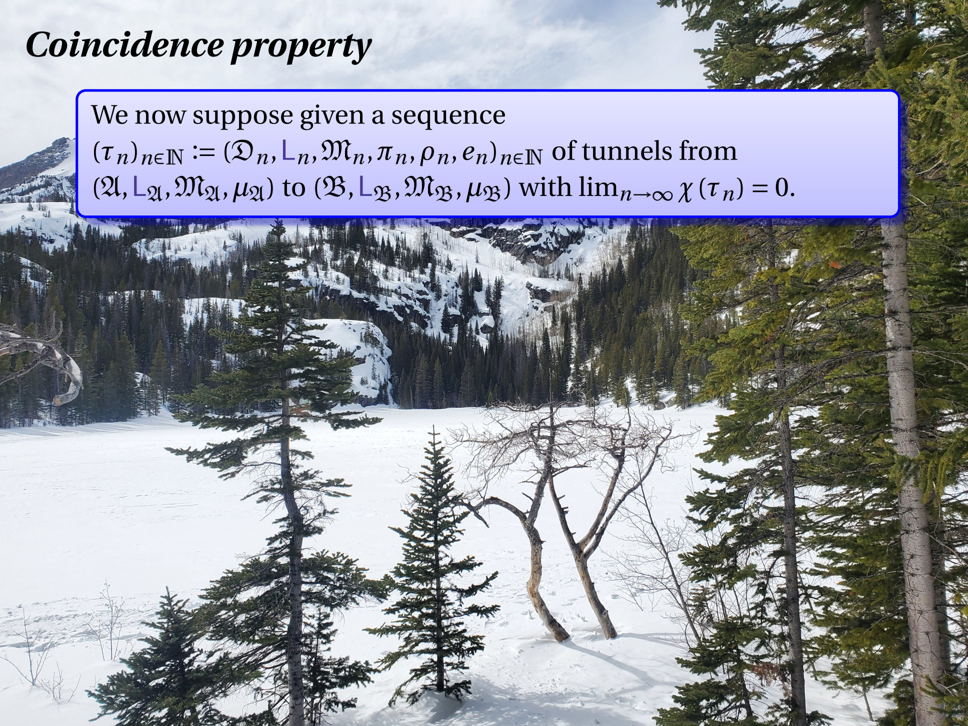
$$\mathfrak{t}_\tau(a|l) := \left\{\rho(d) : d \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}), \|d\|_{\mathsf{L},M} \leqslant l, \pi(d) = a\right\}.$$

Corollary (L. 25)

For all $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L})$, if $l > \|a\|_{\mathsf{L},M}$, then

$$\mathrm{diam}\left(e\mathfrak{t}_\tau(a|l)e, \mathfrak{B}\right) \leqslant 2l\chi(\tau).$$

Moreover, $e\mathfrak{t}_\tau(a|l)e$ is inside $\left\{ebe : \|b\|_{\mathsf{L},M} \leqslant l\right\}$, which is *compact* since $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a proper quantum metric space.



Coincidence property

We now suppose given a sequence $(\tau_n)_{n\in\mathbb{N}} := (\mathfrak{D}_n, \mathsf{L}_n, \mathfrak{M}_n, \pi_n, \rho_n, e_n)_{n\in\mathbb{N}}$ of tunnels from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$ with $\lim_{n\to\infty} \chi(\tau_n) = 0$.

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- If $e_n^\mathfrak{B} := \rho_n(e_n)$ for all $n \in \mathbb{N}$, then $(e_n^\mathfrak{B})$ is a Lipschitz exhaustive sequence (in $\mathfrak{M}_\mathfrak{B}$), hence an approximate unit in $\mathfrak{B}$.
- for each $p \in \mathbb{N}$, $a \in \mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_\mathfrak{A})$ and $l > \|a\|_{\mathsf{L}_\mathfrak{A},M}$, the sequence $(e_p^\mathfrak{B} e_m^\mathfrak{B} \mathfrak{t}_{\tau_m}(a|l) e_m^\mathfrak{B} e_p^\mathfrak{B})_{m\in\mathbb{N}}$ has a convergent subsequence converging to some $\{\beta_p(a)\}$ for $\mathsf{Haus}_\mathfrak{B}$, independent of $l > \|a\|_{\mathsf{L}_\mathfrak{A},M}$.
- Use separability of $\mathfrak{A}$ and the main lemma to extract a sequence from $(\tau_n)_{n\in\mathbb{N}}$ so that $(e_p^\mathfrak{B} e_m^\mathfrak{B} \mathfrak{t}_{\tau_m}(a|l) e_m^\mathfrak{B} e_p^\mathfrak{B})_{m\in\mathbb{N}}$ always converges for $\mathsf{Haus}_\mathfrak{B}$.
- $\left\|\beta_p(a)\right\|_\mathfrak{B} \leqslant \|a\|_\mathfrak{A}$ and $\left\|\beta_p(a)\right\|_{\mathsf{L}_\mathfrak{B},M} \leqslant \|a\|_{\mathsf{L}_\mathfrak{A},M}$ thanks to continuity and lower semicontinuity.

Coincidence property

We now suppose given a sequence $(\tau_n)_{n\in\mathbb{N}} := (\mathfrak{D}_n, \mathsf{L}_n, \mathfrak{M}_n, \pi_n, \rho_n, e_n)_{n\in\mathbb{N}}$ of tunnels from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$ with $\lim_{n\to\infty}\chi(\tau_n) = 0$.

- More complicated: $(\beta_p(a))$ is Cauchy in $\mathfrak{B}$. Issue is that there is no relation between $e_p^{\mathfrak{B}}$ and $e_q^{\mathfrak{B}}$ for $p \neq q$, so need a more complicated path.
- Let $\pi(a)$ be the limit of $(\beta_p(a))_{p\in\mathbb{N}}$: π is our candidate for the restriction of a full quantum isometry to $\mathrm{dom}_{\mathrm{sa}}(\mathsf{L}_{\mathfrak{A}})$.
- It is reasonable to show π is linear. To make π a Jordan-Lie morphism is more involved and relies among other things on Leibniz property.
- π can then be extended by continuity, then linearity, to a *-morphism $\mathfrak{A}$.
- Last, tunnels can be “reversed” to construct an inverse map for π.

The Quantum Metametric

Definition (L. 25)

The *quantum metametric* $\text{Đ}(\mathbb{A}, \mathbb{B})$ between any two pointed proper quantum metric spaces $\mathbb{A}$ and $\mathbb{B}$ is defined by:

$$\boxed{\text{Đ}(\mathbb{A}, \mathbb{B}) := \inf \left\{ \varepsilon > 0 : \sup_{r \leqslant \frac{1}{\varepsilon}} \eth_r(\mathbb{A}, \mathbb{B}) < \varepsilon \right\}.}$$

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Theorem (L. 25)

For all pointed proper quantum metric spaces $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$,

- $\mathsf{Đ}(\mathbb{A},\mathbb{B}) = \mathsf{Đ}(\mathbb{B},\mathbb{A})$,

The Quantum Metametric

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The *quantum metametric* $\mathsf{\unicode{x0110}}(\mathbb{A},\mathbb{B})$ between any two pointed proper quantum metric spaces $\mathbb{A}$ and $\mathbb{B}$ is defined by:

$$\boxed{\mathsf{\unicode{x0110}}(\mathbb{A},\mathbb{B}) := \inf\left\{ \varepsilon > 0 : \sup_{r \leqslant \frac{1}{\varepsilon}} \eth_r(\mathbb{A},\mathbb{B}) < \varepsilon \right\}.}$$

Theorem (L. 25)

For all pointed proper quantum metric spaces $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$,

- $\mathsf{\unicode{x0110}}(\mathbb{A},\mathbb{B}) = \mathsf{\unicode{x0110}}(\mathbb{B},\mathbb{A})$,
- $\mathsf{\unicode{x0110}}(\mathbb{A},\mathbb{B}) \leqslant (1+\mathsf{\unicode{x0110}}(\mathbb{D},\mathbb{B}))^2\,\mathsf{\unicode{x0110}}(\mathbb{A},\mathbb{D}) + (1+\mathsf{\unicode{x0110}}(\mathbb{A},\mathbb{D}))^2\,\mathsf{\unicode{x0110}}(\mathbb{B},\mathbb{D})$.

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For all pointed proper quantum metric spaces $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$,

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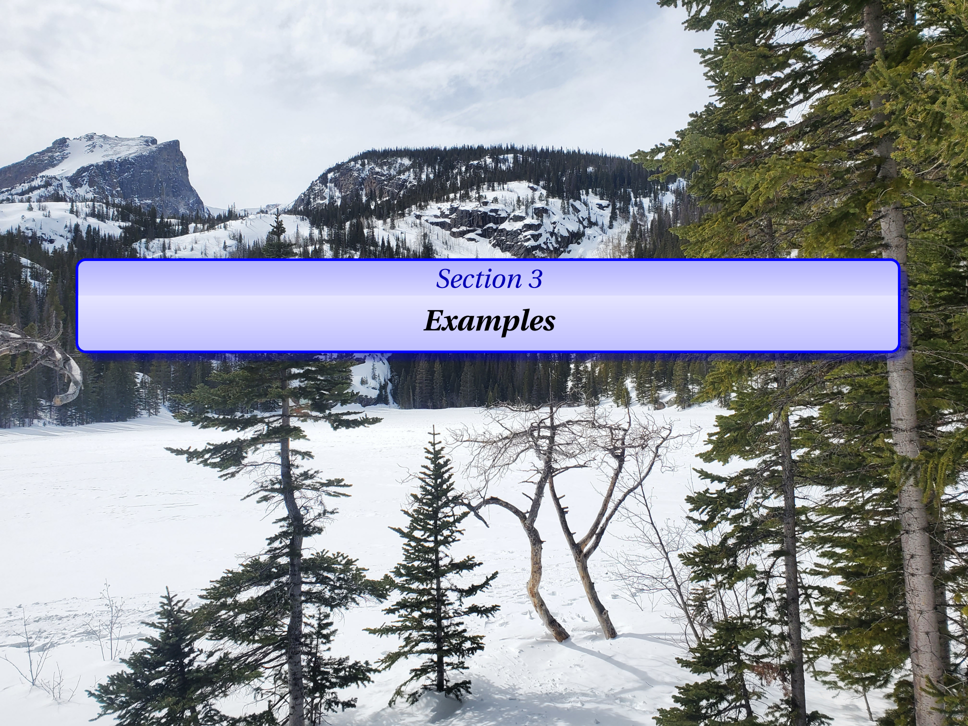
- $\text{Đ}(\mathbb{A},\mathbb{B}) = 0$ if, and only if, there exists a full, pointed quantum isometry from $\mathbb{A}$ to $\mathbb{B}$.



Section 3

Examples

Compact Quantum Metric Spaces I

Theorem (L. 25)

If $(\mathfrak{A}_n, \mathsf{L}_n, \mu_n)_{n\in\mathbb{N}}$ is a sequence of *pointed quantum compact metric spaces* converging for the *pointed propinquity* to $(\mathfrak{A}, \mathsf{L}, \mu)$, then

$$\boxed{\lim_{n\to\infty} \sup_{M\geqslant 1} \eth_M((\mathfrak{A}_n, \mathsf{L}_n, \mathbb{C}1_n, \mu_n), (\mathfrak{A}, \mathsf{L}, \mathbb{C}1_{\mathfrak{A}}, \mu)) = 0,}$$

and in particular,

$$\boxed{\lim_{n\to\infty} \text{Đ}((\mathfrak{A}_n, \mathsf{L}_n, \mathbb{C}1_n, \mu_n), (\mathfrak{A}, \mathsf{L}, \mathbb{C}1_{\mathfrak{A}}, \mu)) = 0.}$$

Compact Quantum Metric Spaces II

Theorem (L. 25)

If $(\mathfrak{A}_n, \mathsf{L}_n, \mu_n)_{n \in \mathbb{N} \cup \{\infty\}}$ is a family of *pointed quantum compact metric spaces*, and if

$$\boxed{\lim_{n\to\infty} \text{Đ}((\mathfrak{A}_n, \mathsf{L}_n, \mathfrak{M}_n, \mu_n), (\mathfrak{A}_\infty, \mathsf{L}_\infty, \mathfrak{M}_\infty, \mu_\infty)) = 0,}$$

for some choice of topography $\mathfrak{M}_n$ of $(\mathfrak{A}_n, \mathsf{L}_n)$ for each $n \in \mathbb{N} \cup \{\infty\}$, then, for $r > \operatorname{diam}(\mathscr{S}(A_\infty), \mathsf{L}_\infty)$,

$$\boxed{\lim_{n\to\infty} \Lambda^{\bullet *}_{2,4r}((\mathfrak{A}_n, \mathsf{L}_n, \mu_n), (\mathfrak{A}_\infty, \mathsf{L}_\infty, \mu_\infty)) = 0.}$$

The Classical Case

Theorem (L. 25)

Let $(X_n, d_n, x_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a family of *pointed proper metric spaces*. If

$$\boxed{\lim_{n\to\infty} \mathsf{GH}((X_n, d_n, x_n), (Y, d_Y, y)) = 0,}$$

then

$$\boxed{\lim_{n\to\infty} \mathsf{Đ}((C_0(X_n), \mathsf{L}_{d_n}, C_0(X_n), \delta_{x_n}), (C_0(X_\infty), \mathsf{L}_{d_\infty}, C_0(X_\infty), \delta_{x_\infty})) = 0.}$$

Example

We let $c(\mathbb{Z}/n) \rtimes_{\cdot + [t]} \mathbb{Z}/n$ act similarly on $\ell^2(\mathbb{Z}/n)$ and set

$$\mathsf{L}_n(a) = \left\| \partial_{\mathbb{Z}}(a)\gamma_1 + \partial_{\mathbb{U}_n}(a)\gamma_2 \right\|,$$

with $\partial_{\mathbb{Z}}(a)$ multiplication by $(a(n+1) - a(n))_{n \in \mathbb{Z}/n}$ and

$$\partial_{\mathbb{U}_n}(a) = [\mathsf{Clock_n}, a]$$

for the Clock matrix $\mathrm{diag}\left(\exp\left(\frac{2ij\pi}{n}\right)\right)_{j \in \mathbb{Z}/n}$, then:

$$\boxed{\begin{aligned}\lim_{n \to \infty} \text{Đ}\Big(& (c_0(\mathbb{Z}) \rtimes \mathbb{Z}, \mathsf{L}, c_0(\mathbb{Z}), \delta_0 \circ \mathbb{E}), \\ & (c(\mathbb{Z}/n) \rtimes \mathbb{Z}/n, \mathsf{L}_n, c(\mathbb{Z}/n), \delta_0 \circ \mathbb{E}_n) \Big) = 0.\end{aligned}}$$