The Spectral Propinquity

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Metric Spectral triples

Definition

A *metric spectral triple* $(\mathfrak{A}, \mathscr{H}, D\!\!\!\!/\,)$ is given by:

1. $\mathfrak{A}$ is a unital C*-algebra of operators acting on a Hilbert space $\mathscr{H}$,
2. $D\!\!\!\!/$ is a self-adjoint operator defined on a dense domain in $\mathscr{H}$ with compact resolvent,
3. $\{a \in \mathfrak{A} : [D\!\!\!\!/, a] \text{ is bounded }\}$ is a dense *-subalgebra of $\mathfrak{A}$,

Metric Spectral triples

Definition

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2. $\not{D}$ is a self-adjoint operator defined on a dense domain in $\mathscr{H}$ with compact resolvent,
3. $\{a \in \mathfrak{A} : [\not{D}, a] \text{ is bounded }\}$ is a dense *-subalgebra of $\mathfrak{A}$,
4. if we set, for all states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$,

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) := \sup\left\{|\varphi(a) - \psi(a)| : |||[\not{D}, a]|||_{\mathscr{H}} \leqslant 1\right\}$$

then mk_{L} is a metric over $\mathscr{S}(\mathfrak{A})$ which induces the weak* topology.

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1. A topology on the space of metric spectral triples
2. Examples
3. Spectral Properties

A topology for metric spectral triples

Our project

We wish to construct a metric, up to unitary equivalence, on the space of metric spectral triples, to formalize such heuristics as:

1. the convergence of matrix models in physics,
2. spectral triples on fractals or AF algebras as limits of simpler spectral triples,
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...

Gromov-Hausdorff distance

$$\begin{tikzcd} & (Z, d_Z) & \\ (X, d_X) \arrow[ur, hook, "j_X"] & & (Y, d_Y) \arrow[ul, hook', "j_Y"'] \end{tikzcd}$$

Figure: j_X, j_Y isometries

$$\begin{aligned} d_{\mathrm{GH}}((X, d_X), (Y, d_Y)) := \inf\Big\{ & d_Z(j_X(X), j_Y(Y)) : \\ & j_X : X \hookrightarrow Z, j_Y : Y \hookrightarrow Z \text{ isometries }, (Z, d_Z) \text{ compact m.s.} \Big\} \end{aligned}$$

Gromov-Hausdorff distance

$$\begin{tikzcd}
 & (C(Z), \mathsf{L}_{d_Z}) \arrow[dl, "f \mapsto f\circ j_Y"'] \arrow[dr, "f \mapsto f\circ j_X"] & \\
(C(X), \mathsf{L}_{d_X}) & & (C(Y), \mathsf{L}_{d_Y})
\end{tikzcd}$$

Figure:

$$\mathsf{L}_{d_Z} : f \in C(Z) \mapsto \sup\left\{ \frac{|f(x) - f(y)|}{d_Z(x,y)} : x, y \in Z, x \neq y \right\}.$$

$$d_{\mathrm{GH}}((X,d_X),(Y,d_Y)) := \inf\Big\{ d_Z(j_X(X), j_Y(Y)) :$$
$$j_X : X \hookrightarrow Z, j_Y : Y \hookrightarrow Z \text{ isometries }, (Z, d_Z) \text{ compact m.s.}\Big\}$$

Gromov-type construction for metric spectral triples

$$\begin{tikzcd} & ? \arrow[dl, "\pi_{\mathfrak{A}}"'] \arrow[dr, "\pi_{\mathfrak{B}}"] & \\ (\mathfrak{A}, \mathscr{H}, \mathbb{D}) & & (\mathfrak{B}, \mathscr{J}, \nabla) \end{tikzcd}$$

Figure:

We begin with the quantum metrics

$$
\begin{tikzcd}
 & (\mathfrak{D}, \mathsf{L}) \arrow[dl, "\rho_{\mathfrak{A}}"'] \arrow[dr, "\rho_{\mathfrak{B}}"] & \\
\left(\mathfrak{A}, {|||[\not{D}, \cdot]|||}_{\mathscr{H}}\right) & & \left(\mathfrak{B}, {|||[\not{\nabla}, \cdot]|||}_{\mathscr{J}}\right)
\end{tikzcd}
$$

Figure: $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$ are quantum isometries

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 & (\mathfrak{D}, \mathsf{L}) \arrow[dl, "\rho_{\mathfrak{A}}"'] \arrow[dr, "\rho_{\mathfrak{B}}"] & \\
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\end{tikzcd}$$

Figure: $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$ are quantum isometries

$\varphi, \psi \in \mathscr{S}(\mathfrak{D}) \mapsto \mathsf{mk}_{\mathsf{L}}(\varphi, \psi) := \sup\left\{|\varphi(d) - \psi(d)| : \mathsf{L}(d) \leqslant 1\right\}$ metrizes $\sigma(\mathfrak{D}^*, \mathfrak{D})$ on $\mathscr{S}(\mathfrak{D})$ and L satisfies a Leibniz property and $\mathsf{L}(1) = 0$.

We begin with the quantum metrics

$$\begin{tikzcd}[column sep=small] & (\mathfrak{D}, \mathsf{L}) \arrow[dl, "\rho_{\mathfrak{A}}"'] \arrow[dr, "\rho_{\mathfrak{B}}"] & \\ \left(\mathfrak{A}, {|||[\not{D}, \cdot]|||}_{\mathscr{H}}\right) & & \left(\mathfrak{B}, {|||[\not{\nabla}, \cdot]|||}_{\mathscr{J}}\right) \end{tikzcd}$$

Figure: $\rho_{\mathfrak{A}}, \rho_{\mathfrak{B}}$ are quantum isometries

$$\chi(\tau) := \max\Big\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}}}\left(\mathscr{S}(\mathfrak{D}), \{\varphi \circ \rho_{\mathfrak{A}} : \varphi \in \mathscr{S}(\mathfrak{A})\}\right), \\ \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}}}\left(\mathscr{S}(\mathfrak{D}), \{\varphi \circ \rho_{\mathfrak{B}} : \varphi \in \mathscr{S}(\mathfrak{B})\}\right)\Big\}.$$

What about the Hilbert spaces?

$$\begin{tikzcd}
 & (\mathscr{E}, \mathsf{T}) \arrow[dl, "\Pi_{\mathfrak{A}}"'] \arrow[dr, "\Pi_{\mathfrak{B}}"] & \\
(\mathscr{H}, \|\cdot\|_{\mathscr{H}} + \|\slashed{D}\cdot\|_{\mathscr{H}}) & & (\mathscr{J}, \|\cdot\|_{\mathscr{J}} + \|\slashed{\nabla}\cdot\|_{\mathscr{J}})
\end{tikzcd}$$

Figure:

What about the scalars?

$$\begin{tikzcd}
 & & (\mathscr{E}, \mathsf{T}) \arrow[ddll, "\Pi_{\mathfrak{A}}"] \arrow[ddrr, "\Pi_{\mathfrak{B}}"'] & & \\
 & & (\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}) \arrow[u] \arrow[ddll, bend left, "\pi_{\mathfrak{A}}"] \arrow[ddrr, bend right, "\pi_{\mathfrak{B}}"'] & & \\
(\mathscr{H}, \|\cdot\|_{\mathscr{H}} + \|\slashed{D}\cdot\|_{\mathscr{H}}) & & & & (\mathscr{J}, \|\cdot\|_{\mathscr{J}} + \|\slashed{\nabla}\cdot\|_{\mathscr{J}}) \\
(\mathbb{C}, 0) \arrow[u] & & & & (\mathbb{C}, 0) \arrow[u]
\end{tikzcd}$$

Figure:

Putting it together

$$
\begin{tikzcd}
 & (\mathfrak{D}, \mathsf{L}) \arrow[d] \arrow[ddl, "\rho_{\mathfrak{A}}"'] \arrow[ddr, "\rho_{\mathfrak{B}}"] & \\
 & (\mathscr{E}, \mathsf{T}) \arrow[ddl, "\Pi_{\mathfrak{A}}"] \arrow[ddr, "\Pi_{\mathfrak{B}}"'] & \\
(\mathfrak{A}, |||[\not{D}, \cdot]|||_{\mathscr{H}}) \arrow[d] & (\mathfrak{E}, \mathsf{L}_{\mathfrak{E}}) \arrow[u] \arrow[ddl, bend left, "\pi_{\mathfrak{A}}"] \arrow[ddr, bend right, "\pi_{\mathfrak{B}}"'] & (\mathfrak{B}, |||[\not{\nabla}, \cdot]|||_{\mathscr{J}}) \arrow[d] \\
(\mathscr{H}, \|\cdot\|_{\mathscr{H}} + \|\not{D}\cdot\|_{\mathscr{H}}) & & (\mathscr{J}, \|\cdot\|_{\mathscr{J}} + \|\not{\nabla}\cdot\|_{\mathscr{J}}) \\
(\mathbb{C}, 0) \arrow[u] & & (\mathbb{C}, 0) \arrow[u]
\end{tikzcd}
$$

Figure:

Distorsion

Definition

$$K_\varepsilon(\xi,\eta) := \sup_{\substack{0\leqslant t\leqslant\frac{1}{\varepsilon}\\ \mathsf{T}(\omega)\leqslant 1}} \left\{ \left| \langle \exp(i t \not{D})\xi, \Pi_{\mathfrak{A}}(\omega)\rangle_{\mathscr{H}} - \langle \exp(i t \nabla\!\!\!/)\eta, \Pi_{\mathfrak{B}}(\omega)\rangle_{\mathscr{J}} \right| \right\}$$

$$\begin{aligned}\mathsf{sep}_\varepsilon(\not{D}, \nabla|\tau) := \mathsf{Haus}_{K_\varepsilon}\Big(&\left\{\xi \in \mathrm{dom}(\not{D}) : \|\xi\|_{\mathscr{H}} + \|\not{D}\xi\|_{\mathscr{H}} \leqslant 1\right\}, \\ &\left\{\eta \in \mathrm{dom}(\nabla\!\!\!/) : \left\|\eta\right\|_{\mathscr{J}} + \left\|\nabla\!\!\!/\eta\right\|_{\mathscr{J}} \leqslant 1\right\}\Big)\end{aligned}$$

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1 A topology on the space of metric spectral triples

2 Examples

3 Spectral Properties

A first collapse result

Theorem (L., 24)

If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a metric spectral triple and $0 \in \mathrm{Sp}(\not{D})$, then

$$\lim_{\varepsilon \to 0} \Lambda^{\mathrm{spec}}\left(\left(\mathfrak{A}, \mathscr{H}, \frac{1}{\varepsilon}\not{D}\right), (\mathbb{C}, \ker \not{D}, 0)\right) = 0.$$

If $0 \notin \mathrm{Sp}(\not{D})$, then $\left(\mathfrak{A}, \mathscr{H}, \frac{1}{\varepsilon}\not{D}\right)_{\varepsilon > 0}$ does not converge for Λ^{spec}.

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Thank you!

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