A Geometry for the space of spectral triples

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The Space of Noncommutative "Drums"

A. Connes introduced in 1985 a generalization of *spectral geometry* to the *noncommutative realm* by means of a structure called a *spectral triple*, i.e. a noncommutative analogue of the Dirac operators on Riemannian spin manifolds.

The Space of Noncommutative Manifolds

A *spectral triple* $\boxed{(\mathfrak{A}, \mathscr{H}, \mathbb{D})}$ is an analogue of a first order pseudo-differential operator (of Dirac type), where

- $\mathfrak{A}$ is a *unital C*-algebra*, i.e. a noncommutative analogue of $C(X)$ for a compact Hausdorff space X,
- $\mathscr{H}$ is a *Hilbert space* on which $\mathfrak{A}$ acts, analogue to the space of sections of some spinor bundles,
- $\mathbb{D}$ is a *self-adjoint operator* densely defined on $\mathscr{H}$ with *compact resolvent* and *bounded commutator* with a dense *-subalgebra of $\mathfrak{A}$; formally

$$\mathfrak{A}_{\mathbb{D}} = \{a \in \mathfrak{A} : a\,\mathrm{dom}\,(\mathbb{D}) \subseteq \mathrm{dom}\,(\mathbb{D}) \text{ and } [\mathbb{D}, a] \text{ is bounded}\}$$

  is a dense *-subalgebra of $\mathfrak{A}$.

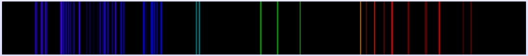

Figure: The spectrum of oxygen

The Space of Noncommutative Manifolds

Spectral triples are *much more flexible* than manifold structure, and can be constructed over *noncommutative C*-algebras*, fractals and other singular spaces, and even finite sets. Much success has been achieved in extending Atiah-Singer index theorem to spectral triples to many such generalized setting.

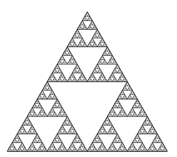

Figure: The Sierpiński Triangle

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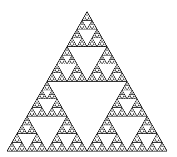

Figure: The Sierpiński Triangle

Spectral triples allow us to extend important notions from physics. The *spectral action* of a spectral triple $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$, in particular, generalizes the Hilbert action of relativity, and is relevant to physics over noncommutative spaces.

Matrix Models: Clock and Shift, Fuzzy Tori

$$C_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_n^{n-1} \end{pmatrix} \text{ and } S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix},$$

with $r_n = \exp\left(\frac{2i\pi}{n}\right)$. Note that $C^*(C_n, S_n) = \mathfrak{B}(\mathbb{C}^n)$.

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with $r_n = \exp\left(\frac{2i\pi}{n}\right)$. Note that $C^*(C_n, S_n) = \mathfrak{B}(\mathbb{C}^n)$. Let $\mathbb{D}_n$ be defined on $\mathscr{H}_n = \mathfrak{B}(\mathbb{C}^n) \otimes \mathbb{C}^4$ be defined by:

$$\begin{aligned} \mathbb{D}_n = \frac{n}{2\pi}\Bigg(\left[\frac{S_n + S_n^*}{2}, \cdot\right] \otimes \gamma_1 + \left[\frac{S_n - S_n^*}{2i}, \cdot\right] \otimes \gamma_2 \\ + \left[\frac{C_n^* + C_n}{2}, \cdot\right] \otimes \gamma_3 + \left[\frac{C_n^* - C_n}{2i}, \cdot\right] \otimes \gamma_4 \Bigg), \end{aligned}$$

with $\gamma_j\gamma_k + \gamma_k\gamma_j = \delta_j^k$ $(j, k \in \{1,2,3,4\})$. Then $(C^*(C_n, S_n), \mathscr{H}_n, \mathbb{D}_n)$ *is a spectral triple*.

A geometry for the space of quantum spaces

The Questions

1. How do we formalize convergence of spectral triples?
2. Can we construct new spectral triples as limits?
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For example: is there a limit for the spectral triples associated with clock/shift matrix models? Are spectral triples over fractals limits of spectral triples over simpler curves? Are spectral triples over certain inductive limits natural limits of spectral triples on the factors?

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1. *Connes' metric* on the state space of $\mathfrak{A}$: Connes' distance between any two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$ is:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) := \sup\left\{|\varphi(a) - \psi(a)| : a\,\mathrm{dom}(\slashed{D}) \subseteq \mathrm{dom}(\slashed{D}), \|[\slashed{D}, a]\|_{\mathscr{H}} \leqslant 1\right\};$$

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A *spectral triple* $(\mathfrak{A}, \mathscr{H}, D)$ induces three structures relevant to our goal:

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The core of our work is noncommutative metric geometry: *metric spectral triples are examples of compact quantum metric spaces*, for which we devised an analogue of the Gromov-Hausdorff distance.

GPS

1. Compact Quantum Metric Spaces
2. The Gromov-Hausdorff Propinquity
3. The Spectral Propinquity
4. Examples and Applications

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\boxed{\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y \right\}}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

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5. $\mathsf{L}\left(\frac{ab+ba}{2}\right) \vee \mathsf{L}\left(\frac{ab-ba}{2i}\right) \leqslant F\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}\right) + K\mathsf{L}(a)\mathsf{L}(b)$;

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We call L an *L-seminorm*.

Some facts about quantum compact metric spaces

- Quantum compact metric spaces form a *category* with arrows given by Lipschitz morphisms: a *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$,

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- Examples include *quantum tori* (Rieffel, 98, 02; L., 21), *AF algebras* (Aguilar-L., 15) and other *inductive limits* (Aguilar-L, 22; Farsi-L-Packer, 23), *Podles spheres* and $SU_q(2)$ (Aguilar-Kaad-Kyed, Kaad-Kyed, 19-22), some *C*-crossed-products* (Hawkins-Skalski-White-Zacharias, 13; Klisse, 23), many *discrete group C*-algebras* (Rieffel-Ozawa, 01; Rieffel-Christ, 15), and *more*.

Metric Spectral Triples

Definition (L., 18)

A spectral triple $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is *metric* when $\mathsf{mk}_{\mathbb{D}}$ is a metric on the state space $\mathscr{S}(\mathfrak{A})$, which induces the *weak* topology*.

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In other words, $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is a metric spectral triple, if and only if, setting:

$$\mathrm{dom}(\mathsf{L}_{\mathbb{D}}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : a \cdot \mathrm{dom}(\mathbb{D}) \subseteq \mathrm{dom}(\mathbb{D}), [\mathbb{D}, a] \text{ bounded}\}$$

and, for all $a \in \mathrm{dom}(\mathsf{L}_{\mathbb{D}})$,

$$\mathsf{L}_{\mathbb{D}}(a) = |||[\mathbb{D}, a]|||_{\mathscr{H}},$$

then $(\mathfrak{A}, \mathsf{L}_{\mathbb{D}})$ is a compact quantum metric space.

The Gromov-Hausdorff Distance

Te *Hausdorff distance* Haus_d between two closed subsets A_1 and A_2 of a compact metric space (X, d) is defined by

$$\mathsf{Haus}_d(A_1, A_2) = \max_{\{j,k\}=\{1,2\}} \sup_{x\in A_j} \inf_{y\in A_k} d(x, y).$$

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Definition (Hausdorff, 1903; Edwards, 75; Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf \left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) \middle| \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\},$$

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The *Gromov-Hausdorff distance* is a *complete metric*, up to isometry, on the class of compact metric spaces.

Quantum Isometries

A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\mathrm{dom}\,(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}\,(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (99), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \qquad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) = \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) = \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Theorem (Rieffel, 99)

If $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a quantum isometry, then $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is an isometry from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ into $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

Quantum Isometries

A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital $*$-morphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a $*$-epimorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and

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Theorem (L., 18)

If $(\mathfrak{A}_1, \mathscr{H}_1, D\!\!\!/_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, D\!\!\!/_2)$ are two *unitarily equivalent metric spectral triples*, then $(\mathfrak{A}_1, \mathsf{L}_{D\!\!\!/_1})$ and $(\mathfrak{A}_2, \mathsf{L}_{D\!\!\!/_2})$ are *fully quantum isometric*.

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd}[column sep=small] & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"] \arrow[dr, "\pi_{\mathfrak{B}}"'] & \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{tikzcd}$$

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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Definition (The extent of a tunnel, L. 13,14)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

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$$\left.\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^*(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

The *dual propinquity* $\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.}$$

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Theorem (L., 13)

The *dual propinquity* Λ^*, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$) by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}}$$

is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"'] \arrow[dr, "\pi_{\mathfrak{B}}"] & \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{tikzcd}$$

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

Some facts about the propinquity

- The set of *classical* compact metric spaces is *nowhere dense* and *closed*.

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Metrical C-correspondences*

If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a *metric spectral triple*, then

$$\boxed{\mathrm{qvb}(\mathfrak{A}, \mathscr{H}, \not{D}) := (\mathscr{H}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\not{D}}, \mathbb{C}, 0)}$$

where $\mathsf{D} = \|\cdot\|_{\mathscr{H}} + \|\not{D}\cdot\|_{\mathscr{H}}$, is an example of the following structure.

Definition (L. (16,18,19))

A *metrical C*-correspondence* $\Omega = (\mathscr{M}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is given by:

1. two *quantum compact metric spaces* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$,
2. an $\mathfrak{A}$-$\mathfrak{B}$ C*-correspondence $\mathscr{M}$, with $\mathfrak{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathscr{M}}$,
3. D is a *norm* on a dense subspace of $\mathscr{M}$ such that:
    1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
    2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
    3. $\forall \eta, \omega \in \mathscr{M} \quad \max\{\mathsf{L}_{\mathfrak{B}}(\Re\langle\omega,\eta\rangle_{\mathscr{M}}), \mathsf{L}_{\mathfrak{B}}(\Im\langle\omega,\eta\rangle_{\mathscr{M}})\} \leqslant H\mathsf{D}(\omega)\mathsf{D}(\eta)$,
    4. $\forall \eta \in \mathscr{M} \quad \forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{D}(a\eta) \leqslant G(\|a\|_{\mathfrak{A}} + \mathsf{L}_{\mathfrak{A}}(b))\mathsf{D}(\eta)$.

Tunnels between Metrical C*-correspondeces

$$
\begin{tikzcd}
 & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] & \\
(\mathfrak{A}_1, \mathsf{L}_1) & & (\mathfrak{A}_2, \mathsf{L}_2)
\end{tikzcd}
$$

A tunnel: $\mathsf{L}_j(a) = \inf \mathsf{L}_{\mathfrak{D}}(\theta_j^{-1}(\{a\}))$.

Tunnels between Metrical C-correspondeces*

$$\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[d, dashed] \arrow[ddll, "\theta_1"'] \arrow[ddrr, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{M}_1, \mathsf{D}_1) & & (\mathscr{M}_2, \mathsf{D}_2) & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"]
\end{tikzcd}$$

$\mathscr{J}$ is a $\mathfrak{D}$-module, $\mathsf{D}_j(\omega) = \inf \mathsf{T}(\Pi_j^{-1}(\{\omega\}))$, T D-norm

Tunnels between Metrical C*-correspondeces

$$\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[d, dashed] \arrow[dddll, bend right, "\theta_1"'] \arrow[dddrr, bend left, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[d, dotted] \arrow[ddl, bend right, "\Pi_1"'] \arrow[ddr, bend left, "\Pi_2"] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[ddl, "\pi_1"] \arrow[ddr, "\pi_2"'] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{M}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{M}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathfrak{B}_1, \mathsf{S}_1) & & (\mathfrak{B}_2, \mathsf{S}_2) &
\end{tikzcd}$$

$\mathscr{J}$ is a $\mathfrak{D}$-$\mathfrak{C}$-C*-corr; $(\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)$ tunnel.

Extent of Metrical Tunnels

$$
\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[d, dashed] \arrow[dddll, bend right, "\theta_1"'] \arrow[dddrr, bend left, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[d, dotted] \arrow[ddl, bend right, "\Pi_1"'] \arrow[ddr, bend left, "\Pi_2"] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{M}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{M}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathfrak{B}, \mathsf{S}_1) & & (\mathfrak{B}_2, \mathsf{S}_2) &
\end{tikzcd}
$$

$$\chi(\tau) = \max\left\{\chi\left((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2)\right), \chi\left((\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)\right)\right\}.$$

The metrical Propinquity

Definition (L. 16,18)

The *metrical propinquity* $\Lambda^{\mathsf{met}}(\mathbb{A}_1, \mathbb{A}_2)$ between two metrical C*-correspondences $\mathbb{A}_1$ and $\mathbb{A}_2$, is defined by

$$\inf\left\{\chi(\tau) : \tau \text{ is a tunnel from } \mathbb{A}_1 \text{ to } \mathbb{A}_2\right\}.$$

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A nontrivial example of convergence of modules for our metric is given by *Heisenberg modules over quantum tori*, where the D-norm arises from *Connes' connection*.

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When applied to spectral triples, the metrical propinquity provides metric information, but, maybe most importantly, it encodes the *convergence of the domains of the Dirac operators.*

Phase

The phase of a Dirac operator

What do we need to add to our construction of our metric on spectral triples so that distance 0 means equivalence by conjugation?

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If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a spectral triple then:

$$\boxed{t \in \mathbb{R} \mapsto U_t = \exp(i\, t \not{D})}$$

is a strongly continuous action of $\mathbb{R}$ on $\mathscr{H}$ such that:

$$\forall t \in \mathbb{R}, \xi \in \mathscr{H} \quad \mathsf{D}(U_t \xi) = \|U_t \xi\|_{\mathscr{H}} + \|\not{D} U_t \xi\|_{\mathscr{H}} = \mathsf{D}(\xi).$$

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Next step

How do we incorporate convergence of group actions?

Separation for Family of Operators

Let $(\mathscr{M}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a metrical C*-correspondence.

Separation for Family of Operators

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If $\varphi, \psi \in \mathscr{M}^*$, then we set

$$\mathsf{mk}_{\mathsf{D}}(\varphi, \psi) := \sup\left\{|\varphi(\omega) - \psi(\omega)| : \omega \in \mathrm{dom}\,(\mathsf{D}), \mathsf{D}(\omega) \leqslant 1\right\}.$$

If J is any index set, we then set

$$\mathsf{MK}_{\mathsf{D}}((\varphi_j)_{j\in J}, (\psi_j)_{j\in J}) = \sup_{j\in J} \mathsf{mk}_{\mathsf{D}}(\varphi_j, \psi_j).$$

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$\mathscr{S}(\mathbb{M})$ consists of the maps of the form $\zeta \in \mathscr{M} \mapsto \varphi(\langle\zeta, \omega\rangle)$ for $\omega \in \mathscr{M}$ with $\mathsf{D}(\omega) \leqslant 1$, and $\varphi \in \mathscr{S}(\mathfrak{B})$.

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Let now $\tau : \mathbb{A} \xleftarrow{(\Pi_{\mathbb{A}},\ldots} \mathbb{M} \xrightarrow{(\Pi_{\mathbb{B}},\ldots)} \mathbb{B}$ be a tunnel between metrical C*-correspondences.

Definition (L., 23)

Let $J \neq \emptyset$ The *separation* $\mathsf{sep}(A, B|\tau)$ of a family $A := (a_j)_{j\in J}$ in $\mathfrak{B}(\mathbb{A})$, and a family $B := (b_j)_{j\in J}$ in $\mathfrak{B}(\mathbb{B})$, wrt τ, is

$$\mathsf{Haus}_{\mathbb{MK}_{\mathsf{D}}}\left(\left\{\varphi \circ a_j \circ \Pi_{\mathfrak{A}} : \varphi \in \mathscr{S}(\mathbb{A})\right\}, \left\{\psi \circ b_j \circ \Pi_{\mathfrak{B}} : \psi \in \mathscr{S}(\mathbb{B})\right\}\right)$$

The Spectral Propinquity

Definition (L., 18)

The *spectral propinquity* $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1),(\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2))$ between two metric spectral triples $(\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1)$ and $(\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2)$ is

$$\inf\left\{\frac{\sqrt{2}}{2},\varepsilon>0 : \exists\tau \text{ tunnel from } (\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1) \text{ to } (\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2)\right.$$
$$\left.\text{such that } \max\left\{\chi(\tau),\mathsf{sep}\left((e^{it\mathbb{D}_1})_{0\leqslant t\leqslant\frac{1}{\varepsilon}},(e^{it\mathbb{D}_2})_{0\leqslant t\leqslant\frac{1}{\varepsilon}}\middle|\tau\right)\right\}\leqslant\varepsilon\right\}.$$

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$$\left.\text{such that } \max\left\{\chi(\tau),\mathsf{sep}\left((e^{it\mathbb{D}_1})_{0\leqslant t\leqslant\frac{1}{\varepsilon}},(e^{it\mathbb{D}_2})_{0\leqslant t\leqslant\frac{1}{\varepsilon}}|\tau\right)\right\}\leqslant\varepsilon\right\}.$$

Theorem (L., 18)

The *spectral propinquity* Λ^{spec} is a *metric* on the class of spectral triples, up to unitary equivalence, i.e. $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1),(\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2)) = 0$ if, and only if there exists a *unitary* $U:\mathscr{H}_1\to\mathscr{H}_2$ such that $U\mathrm{dom}(\mathbb{D}_1)=\mathrm{dom}(\mathbb{D}_2)$,

$$U\mathbb{D}_1U^* = \mathbb{D}_2 \text{ and } \mathrm{Ad}_U \text{ *-isomorphism from } \mathfrak{A}_1 \text{ to } \mathfrak{A}_2.$$

Matrix Models: Clock and Shift, Fuzzy Tori

$$C_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_n^{n-1} \end{pmatrix} \text{ and } S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

with $r_n = \exp\left(\frac{2i\pi}{n}\right)$. Note that $C^*(C_n, S_n) = \mathfrak{B}(\mathbb{C}^n)$.

Matrix Models: Clock and Shift, Fuzzy Tori

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with $r_n = \exp\left(\frac{2i\pi}{n}\right)$. Note that $C^*(C_n, S_n) = \mathfrak{B}(\mathbb{C}^n)$. Let $\mathbb{D}_n$ be defined on $\mathscr{H}_n = \mathfrak{B}(\mathbb{C}^n) \otimes \mathbb{C}^4$ be defined by:

$$\mathbb{D}_n = \frac{n}{2\pi}\Bigg(\left[\frac{S_n + S_n^*}{2}, \cdot\right] \otimes \gamma_1 + \left[\frac{S_n - S_n^*}{2i}, \cdot\right] \otimes \gamma_2 + \left[\frac{C_n^* + C_n}{2}, \cdot\right] \otimes \gamma_3 + \left[\frac{C_n^* - C_n}{2i}, \cdot\right] \otimes \gamma_4 \Bigg),$$

with $\gamma_j\gamma_k + \gamma_k\gamma_j = \delta_j^k$ $(j, k \in \{1,2,3,4\})$. Then *$(C^*(C_n, S_n), \mathscr{H}_n, \mathbb{D}_n)$ is a spectral triple*.

Convergence of Fuzzy tori

Theorem (L., 21)

The sequence $\left(C^*(C_n, S_n), \mathfrak{B}(\mathbb{C}^n)\otimes\mathbb{C}^4, \not{D}_n\right)_{n\in\mathbb{N}}$, where

$$\not{D}_n = \frac{n}{2\pi}\left(\left[\frac{S_n+S_n^*}{2},\cdot\right]\otimes\gamma_1 + \left[\frac{S_n-S_n^*}{2i},\cdot\right]\otimes\gamma_2 \right.$$
$$\left. + \left[\frac{C_n^*+C_n}{2},\cdot\right]\otimes\gamma_3 + \left[\frac{C_n^*-C_n}{2i},\cdot\right]\otimes\gamma_4\right),$$

converges, for the spectral propinquity, to the spectral triple $(C(T^2), L^2(\mathbb{T}^2)\otimes\mathbb{C}^4, \not{D}_{\mathbb{T}^2})$, where $C(\mathbb{T}^2) = \left\{\left(e^{i\theta}, e^{i\psi}\right) : \theta,\psi\in[0,2\pi]\right\}$, and on a dense subspace of $L^2(\mathbb{T}^2)\otimes\mathbb{C}^4$, we set

$$\not{D}_{\mathbb{T}^2} = \cos(\psi)\partial_\theta\otimes\gamma_1 + \sin(\psi)\partial_\theta\otimes\gamma_2 + \cos(\theta)\partial_\psi\otimes\gamma_3 + \sin(\theta)\partial_\psi\otimes\gamma_4.$$

Convergence of Fuzzy tori

Theorem (L., 21)

The sequence $\left(C^*(C_n, S_n), \mathfrak{B}(\mathbb{C}^n) \otimes \mathbb{C}^4, \not{D}_n\right)_{n\in\mathbb{N}}$, where

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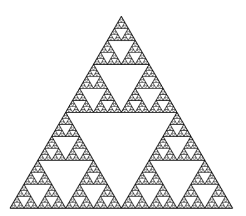
converges, for the spectral propinquity, to the spectral triple $(C(T^2), L^2(\mathbb{T}^2) \otimes \mathbb{C}^4, \not{D}_{\mathbb{T}^2})$, where $C(\mathbb{T}^2) = \left\{\left(e^{i\theta}, e^{i\psi}\right) : \theta, \psi \in [0, 2\pi]\right\}$, and on a dense subspace of $L^2(\mathbb{T}^2) \otimes \mathbb{C}^4$, we set

$$\not{D}_{\mathbb{T}^2} = \cos(\psi)\partial_\theta \otimes \gamma_1 + \sin(\psi)\partial_\theta \otimes \gamma_2 + \cos(\theta)\partial_\psi \otimes \gamma_3 + \sin(\theta)\partial_\psi \otimes \gamma_4.$$

This result is extended to more general fuzzy/quantum tori (L., 21).

Piecewise C^1 Fractals Curves

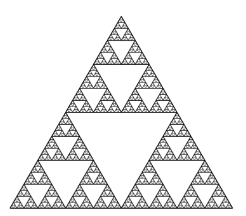


The Sierpińsky gasket

Let $\boxed{\mathscr{J} = \oplus_{n=1}^{\infty} \oplus_{j=1}^{3^n} L^2([-1,1])}$ and $\boxed{\partial(\xi) = \left(2^n \xi'_{n,j}\right)_{n\in\mathbb{N}, j\in\{1,\ldots,3^n\}}}$.

For $n \in \mathbb{N} \setminus \{0\}$, let $C_{n,1}, \ldots, C_{n,3^n}$ be affine functions from $[0,1]$, which parametrize every edge of the level n triangles in $\mathscr{S}\mathscr{G}_n$.

Piecewise C^1 Fractals Curves



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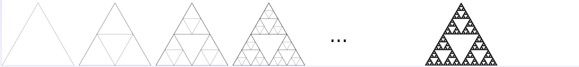
For $n \in \mathbb{N} \setminus \{0\}$, let $C_{n,1},\ldots,C_{n,3^n}$ be affine functions from $[0,1]$, which parametrize every edge of the level n triangles in $\mathscr{S}\mathscr{G}_n$. For $f \in C(\mathscr{S}\mathscr{G}_\infty)$, we set $\boxed{f \cdot \xi_{n,j} := f \circ C_{n,j}(|\cdot|)\xi_{n,j}}$.

$\left(C(\mathscr{S}\mathscr{G}_\infty), \mathscr{J}, \partial\right)$ is a spectral triple, constructed by *Lapidus et al*.

Convergence of certain fractals



For each $n \in \mathbb{N}$, let ∂_n be the restriction of ∂ to $\mathscr{J}_N := \oplus_{n=1}^{N} \oplus_{j=1}^{3^N} L^2([-1,1])$, and let $C(\mathscr{S}\mathscr{G}_N)$ acts on $\mathscr{J}_N$.

Theorem (Landry,Lapidus,L., 20)

Let $(C(\mathscr{S}\mathscr{G}_\infty), \mathscr{J}, \partial)$ be the spectral triple over the Sierpiński gasket $\mathscr{S}\mathscr{G}_\infty$, and for each $n \in \mathbb{N}$, let $(\mathscr{S}\mathscr{G}_n, \mathscr{J}_n, \partial_n)$ be its "restriction" to the finite graph $\mathscr{S}\mathscr{G}_n$. Then

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((C(\mathscr{S}\mathscr{G}_\infty), \mathscr{J}, \partial), (C(\mathscr{S}\mathscr{G}_n), \mathscr{J}_n, \partial_n)) = 0.$$

Collapse!

Theorem (L., 23)

If $(\mathfrak{A}, \mathscr{H}, \mathcal{D})$ is a spectral triple with $0 \in \operatorname{Sp}(\mathcal{D})$, then

$$\lim_{\lambda \to \infty} \Lambda^{\mathrm{spec}}((\mathfrak{A}, \mathscr{H}, \lambda \mathcal{D}), (\mathbb{C}, \ker \mathcal{D}, 0)) = 0.$$

Inductive Limits of Group C-algebras*

Theorem (Farsi-L-Packer, 23)

Let $G = \bigcup_{n\in\mathbb{N}} G_n$ be an Abelian discrete group, with $(G_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of subgroups of G. Let σ be a 2-cocycle of G. Let $\mathbb{L}_H$ be a length function over G such that $\lim_{n\to\infty} \mathsf{Haus}_{\mathbb{L}_H}(G_n, G) = 0$ and $\mathbb{L}_H$ is proper on each G_n. If there exists a strictly increasing function $\mathrm{scale} : \mathbb{N} \to \mathbb{R}$ such that, setting $\mathbb{F} : g \in G \mapsto \mathrm{scale}(\min\{n : g \in G_n\})$, the length $\max\{\mathbb{L}_H, \mathbb{F})$ has the *doubling property*, then

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((C^*(G,\sigma), \ell^2(G)\otimes\mathbb{C}^2, \not{D}), \\ (C^*(G_n,\sigma), \ell^2(G_n)\otimes\mathbb{C}^2, \not{D}_{|\ell^2(G_n)\otimes\mathbb{C}^2})) = 0$$

where

$$\not{D} := \mathrm{mul}_{\mathbb{L}_H} \otimes \gamma_1 + \mathrm{mul}_{\mathbb{F}} \otimes \gamma_2.$$

Applications: noncommutative solenoids, Bunce-Deddens algebras.

Convergence of the Spectrum

Theorem (L., 23)

If $(\mathfrak{A}_n, \mathscr{H}_n, \not{D}_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples converging to $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \not{D}_\infty)$ for the spectral propinquity, then:

$$\boxed{\mathrm{Sp}\left(\not{D}_\infty\right) = \left\{\lambda \in \mathbb{R} : \forall_{\mathbb{N}} n \quad \exists \lambda_n \in \mathrm{Sp}\left(\not{D}_n\right) \quad \lambda = \lim_{n\to\infty} \lambda_n\right\}.}$$

Moreover, under reasonable assumptions, the *multiplicity of the eigenvalues converge as well.*

Convergence of multiplicities

Theorem (L., 23)

If $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples *converging, for the spectral propinquity,* to a metric spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$, and if $\lambda \in \mathrm{Sp}(\mathbb{D}_\infty)$, such that:

1. there exists $\delta > 0$ and $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n > N$, the intersection $\mathrm{Sp}(\mathbb{D}_n) \cap (\lambda - \delta, \lambda + \delta)$ is a *singleton* $\{\lambda_n\}$,
2. if $(\mathrm{multiplicity}(\lambda_n | \mathbb{D}_n))_{n\in\mathbb{N}}$ is *bounded*,

then

$$\boxed{\lim_{n\to\infty} \mathrm{multiplicity}(\lambda_n | \mathbb{D}_n) = \mathrm{multiplicity}(\lambda | \mathbb{D}_\infty).}$$

Convergence of multiplicities

Theorem (L., 23)

If $(\mathfrak{A}_n, \mathscr{H}_n, D\!\!\!/_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples *converging, for the spectral propinquity,* to a metric spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, D\!\!\!/_\infty)$, and if $\lambda \in \mathrm{Sp}(D\!\!\!/_\infty)$, such that:

1. there exists $\delta > 0$ and $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n > N$, the intersection $\mathrm{Sp}(D\!\!\!/_n) \cap (\lambda - \delta, \lambda + \delta)$ is a *singleton* $\{\lambda_n\}$,
2. if $(\mathrm{multiplicity}(\lambda_n | D\!\!\!/_n))_{n\in\mathbb{N}}$ is *bounded*,

then

$$\boxed{\lim_{n\to\infty} \mathrm{multiplicity}(\lambda_n | D\!\!\!/_n) = \mathrm{multiplicity}(\lambda | D\!\!\!/_\infty).}$$

From this, we can conclude, under similar assumptions, that $\mathrm{trace}(f(D\!\!\!/_n))$ converges to $\mathrm{trace}(f(D\!\!\!/))$ for any $f \in C_0(\mathbb{R})$; thus we obtain convergence for spectral actions.

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411.
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *The dual modular propinquity and completeness*, F. Latrémolière, *J. Noncomm. Geometry* **15** (2021) no. 1, 347–398.
- *Metric approximations of spectral triples on the Sierpiński gasket and other fractal curves*, T. Landry, M. Lapidus, F. Latrémolière, *Adv. Math.* **385** (2021), paper no. 107771, 43 pp.
- *Convergence of Spectral Triples on Fuzzy Tori to Spectral Triples on Quantum Tori*, F. Latrémolière, *Comm. Math. Phys.* **388** (2021) no. 2, 1049–1128.
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- *Continuity of the spectrum of Dirac operators of spectral triples for the spectral propinquity*, F. Latrémolière, *Math. Ann.* (2023), https://doi.org/10.1007/s00208-023-02659-x