The Gromov-Hausdorff Propinquity

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Noncommutative metric geometry

Founding Allegory of Noncommutative Metric Geometry

Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

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Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

- Pioneered by *Rieffel (1998–)*, inspired by *Connes (1989)*.
- Motivated by mathematical physics, addresses problems such as:
  - Can we approximate quantum spaces with finite dimensional algebras?
  - Are certain functions from a topological space to quantum spaces continuous? Lipschitz?
  - Are certain functions from a topological space to modules over quantum spaces continuous?

GPS

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\boxed{\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y \right\}}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\boxed{\mathsf{mk}_{\mathsf{L}}(\mu, \nu) = \sup\left\{ \left| \int_X f \, d\mu - \int_X f \, d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.}$$

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

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6. L is lower semi-continuous wrt $\|\cdot\|_{\mathfrak{A}}$.

We call L an *L-seminorm*.

Quantum Tori as Compact Quantum Metric Spaces

Theorem (Rieffel, 98)

If α is a strongly continuous *action* of a *compact group* G, if ℓ is a continuous *length* function over G, and if for all $a \in \mathfrak{A}$ we set:

$$\boxed{\mathsf{L}(a) = \sup\left\{\frac{\|\alpha^g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\}\right\},}$$

then $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space* if and only if $\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$.

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Example

- *Quantum tori* are quantum compact metric spaces with $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$, $G = \mathbb{T}^d$ and ℓ any continuous length on $\mathbb{T}^d$;
- *Noncommutative solenoids* $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]^2, \sigma\right)$ with $G = \mathscr{S}_p^2$ (L., Packer, 16).

AF algebras

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Theorem (Aguilar, L., 15)

If $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ with $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ an increasing sequence of *finite dimensional subalgebras*, if $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$ and if τ is a faithful tracial state on $\mathfrak{A}$, and if for all $a \in \mathfrak{A}$ we set:

$$\boxed{\mathsf{L}(a) = \sup\left\{ \frac{\|\mathbb{E}_n(a) - a\|_{\mathfrak{A}}}{(\dim \mathfrak{A}_n)^{-1}} : n \in \mathbb{N} \right\}}$$

where $\mathbb{E}_n : \mathfrak{A} \twoheadrightarrow \mathfrak{A}$ is the unique *conditional expectation from* $\mathfrak{A}$ *onto* $\mathfrak{A}_n$ such that $\tau \circ \mathbb{E}_n = \tau$ for all $n \in \mathbb{N}$, then $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space.

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These examples include UHF algebras, Effros-Shen algberas, as well as commutative AF algbras such as C(Cantor set).

Certain Spectral Triples

If $(\mathfrak{A}, \mathscr{H}, D)$ is a spectral triple, then a natural candidate for an L-seminorm is given by:

$$\forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{L}_D(a) = |||[D, a]|||_{\mathscr{H}}$$

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Example: Connes, 89

If M is a connected compact spin manifold and D is the Dirac operator of M acting on the Hilbert space $\mathscr{H} = L^2(\Gamma S)$ of square integrable sections of the spinor bundle, then $(C(M), \mathscr{H}, D)$ is a spectral triple such that $\mathsf{mk}_{\mathsf{L}_D}$ restricts to the geodesic distance on M.

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Problem

Some spectral triples are known to produce a Monge-Kantorovich metric which does not to even give finite radius to the state space!

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Theorem (Ozawa, Rieffel, 02)

Let G be a finitely generated group and l a word length function. On $\ell^2(G)$, we define the (unbounded) operator D as multiplication by l.

For all $a \in C^*_{\mathrm{red}}(G)$, set:

$$\boxed{\mathsf{L}_D(a) = |||[D, a]|||_{\ell^2(G)}.}$$

If G is word-hyperbolic then $\left(C^*_{\mathrm{red}}(G), \mathsf{L}\right)$ is a Leibniz quantum compact metric space.

Some other examples

- *Li, 03*: Connes-Landi-du Bois-Voilette's spheres wrt spectral triple,
- *Hawkins, Skalski, White, Zacharias, 13*: C*-crossed products for equicontinuous actions wrt spectral triple,
- *L., 16*: Curved quantum tori, with metric from spectral triple,
- *L., 16*: Perturbations (conformal and otherwise) of metric spectral triples,
- *Christ, Rieffel, 16*: Nilpotent group C*-algebras with Connes' length spectral triple,
- *Aguilar, Kadd, 17*: Podlès spheres

Lipschitz morphisms

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $(\exists k)$ $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $(\exists k)$ $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
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Lipschitz morphisms are arrows for a category over quasi-Leibniz quantum compact metric spaces.

Quantum Isometries

If $X \subset (Z, \mathsf{m})$ and $f \in \mathfrak{sa}(C(X))$:

- if $g \in C(Z)$ and $g\restriction_X = f$, the $\mathrm{Lip}(f) \leqslant \mathrm{Lip}(g)$,
- there exists $g \in \mathfrak{sa}(C(Z))$ with $g\restriction_X = f$ and $\mathrm{Lip}(f) = \mathrm{Lip}(g)$ (McShane, 34).

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Definition (Rieffel (98), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that:

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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The dual maps of quantum isometries are indeed isometries for the Monge-Kantorovich metric.

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \middle| \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Edwards, 75; Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

Tunnels

$$\begin{tikzcd}
 & (Z, \mathsf{m}_Z) & \\
(X, \mathsf{m}_X) \arrow[ur, hook, "\iota_X"] & & (Y, \mathsf{m}_Y) \arrow[ul, hook', "\iota_Y"']
\end{tikzcd}$$

Figure: Gromov-Hausdorff Isometric Embeddings

$$\begin{tikzcd}
 & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, two heads, "\pi_{\mathfrak{A}}"] \arrow[dr, two heads, "\pi_{\mathfrak{B}}"'] & \\
(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})
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Figure: A tunnel

Legend: $\hookrightarrow$: isometry ; $\twoheadrightarrow$: quantum isometry

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Figure: A tunnel

If $\pi : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ then set $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \hookrightarrow \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$.

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(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}) \arrow[ur, hook, "\pi^*_{\mathfrak{A}}"] & & (\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}) \arrow[ul, hook', "\pi^*_{\mathfrak{B}}"']
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Figure: Gromov-Hausdorff Isometric Embeddings

The Dual Gromov-Hausdorff Propinquity

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Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Definition *(The extent of a tunnel, L. 13,14)*

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\boxed{\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{E}}^{*}(\mathscr{S}(\mathfrak{E}))\right) : \mathfrak{E} \in \{\mathfrak{A}, \mathfrak{B}\}\right\}.}$$

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Definition (The Dual Propinquity, L. 13, 14)

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$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{E}}^{*}(\mathscr{S}(\mathfrak{E}))\right) : \mathfrak{E} \in \{\mathfrak{A}, \mathfrak{B}\}\right\}.$$

The *dual propinquity* $\Lambda_F^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.}$$

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd}
& (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"'] \arrow[dr, "\pi_{\mathfrak{B}}"] & \\
(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})
\end{tikzcd}$$

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Theorem (L., 13)

The *dual propinquity* Λ^*_F, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ by:

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is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"'] \arrow[dr, "\pi_{\mathfrak{B}}"] & \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{tikzcd}$$

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

GPS

Examples: Quantum and Fuzzy Tori

Example:

Let ℓ be a *continuous length function* on $\mathbb{T}^d$. For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of $\widehat{G}$, for any $a \in C^*(\widehat{G}, \sigma)$, set:

$$\mathsf{L}_{G,\sigma}(a) = \sup\left\{ \frac{\|\alpha^g(a) - a\|_{C^*(\widehat{G},\sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where α is the dual action of G on $C^*(\widehat{G}, \sigma)$.

Rieffel showed in 1998 that $\left(C^*(\widehat{G}, \sigma), \mathsf{L}_{G,\sigma}\right)$ is a Leibniz quantum compact metric space.

Examples: Quantum and Fuzzy Tori

Example: L., 13

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where α is the dual action of G on $C^*(\widehat{G},\sigma)$.
If $(G_n)_{n\in\mathbb{N}}$ is a *sequence of closed subgroups* of $\mathbb{T}^d$ *converging* to $\mathbb{T}^d$ for the *Hausdorff distance* Haus_ℓ, and if $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of multipliers of $\mathbb{Z}^d$ converging pointwise to some σ, with $\sigma_n(g) = 1$ if g is the coset of 0 for $\widehat{G_n}$, then:

$$\lim_{n\to\infty} \Lambda^*\left(\left(C^*(\widehat{G_n},\sigma_n), \mathsf{L}_{\widehat{G_n},\sigma_n}\right), \left(C^*(\mathbb{Z}^d,\sigma), \mathsf{L}_{\mathbb{Z}^d,\sigma}\right)\right) = 0.$$

Finite Dimensional Approximations of quantum tori

Example: L. (13)

If for all $n \in \mathbb{N}$, we set $\mathscr{F}_n = C^*(U_n, V_n) = C^*(\mathbb{Z}_n^2, \rho_n)$ where:

$$U_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2ip_n\pi}{n}} & & & \\ & & e^{\frac{4ip_n\pi}{n}} & & \\ & & & \ddots & \\ & & & & e^{\frac{2i(n-1)p_n\pi}{n}} \end{pmatrix}$$

with $p_n \not\equiv 0 \mod n$, and if $\lim_{n\to\infty} \frac{p_n}{n} = \theta$, then:

$$\lim_{n\to\infty} \Lambda((\mathscr{F}_n, \mathsf{L}_n), (\mathscr{A}_\theta, \mathsf{L}_\theta)) = 0$$

where $\mathscr{A}_\theta = C^*(U, V)$ and U, V are universal unitaries such that $VU = e^{2i\pi\theta} UV$, while L_n and L are L-seminorms from the dual actions, and for some *fixed* continuous length function on $\mathbb{T}^2$.

Examples: AF algebras

Example: Aguilar, L. 15

Let $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$, $\forall n \in \mathbb{N} \quad \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ and $\dim_{\mathbb{C}} \mathfrak{A}_n < \infty$, such that there exists a *faithful, tracial state* τ on $\mathfrak{A}$. Let $\mathbb{E}_n : \mathfrak{A} \to \mathfrak{A}_n$ for all $n \in \mathbb{N}$ is the *conditional expectation* from $\mathfrak{A}$ onto $\mathfrak{A}_n$ with $\tau \circ \mathbb{E}_n = \tau$.

Set, for all $a \in \mathfrak{A}$:

$$\boxed{\mathsf{L}(a) = \sup\left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{(\dim \mathfrak{A}_n)^{-1}} : n \in \mathbb{N} \right\}.}$$

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Then:

- $(\mathfrak{A}, \mathsf{L}) = \Lambda^* - \lim_{n\to\infty} (\mathfrak{A}_n, \mathsf{L})$,
- the natural map from the Baire space to UHF algebras is Lipschitz.

Effros-Shen AF algebras

Example: Aguilar, L., 15

For $\theta \in (0,1) \setminus \mathbb{Q}$, let $\theta = \lim_{n\to\infty} \frac{p_n^\theta}{q_n^\theta}$ with $\frac{p_n^\theta}{q_n^\theta} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$ for $a_1, \ldots \in \mathbb{N}$.

- Set $\mathfrak{AF}_\theta = \varinjlim_{n\to\infty} \left(\mathfrak{M}_{q_n} \oplus \mathfrak{M}_{q_{n-1}}, \psi_{n,\theta}\right)$ where $\psi_{n,\theta}$ involves a_{n+1}.
- let L_θ the L-seminorm for this data.

For all $\theta \in (0,1) \setminus \mathbb{Q}$, we have:

$$\lim_{\substack{\vartheta\to\theta \\ \vartheta\notin\mathbb{Q}}} \Lambda\left((\mathfrak{AF}_\vartheta, \mathsf{L}_\vartheta), (\mathfrak{AF}_\theta, \mathsf{L}_\theta)\right) = 0.$$

Examples

Example

- *Rieffel, 15* has constructed full matrix approximations for the propinquity of $C(S^2)$ using metrics arising from actions of $SU(2)$.
- *L. and Packer, 16* proved continuity for the noncommutative solenoids as function of the multipliers of $\mathbb{Z}[p^{-1}]^2$.
- *L., 15* proved continuity for curved quantum tori.
- *L., 15* proved continuity for conformal deformations.
- *L., 15* proved that nuclear quasi-diagonal quantum compact metric spaces are limits of finite dimensional quantum compact metric spaces,
- and more …

Compactness

Definition (L., 15)

If $(\mathfrak{A}, \mathsf{L})$ is an F-quantum compact metric space and if $\varepsilon > 0$ then

$$\boxed{\operatorname{cov}_F(\mathfrak{A}, \mathsf{L}|\varepsilon) = \min\left\{\dim \mathfrak{B} : \Lambda^*_F((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \varepsilon\right\}.}$$

Theorem (L., 15)

A class $\mathscr{C}$ of F-quantum compact metric spaces with finite covering numbers is *precompact* for Λ^*_F if and only if:

1. $\exists M > 0 \quad \forall (\mathfrak{A}, \mathsf{L}) \in \mathscr{C} \quad \operatorname{diam}(\mathfrak{A}, \mathsf{L}) \leqslant M$,
2. $\exists G : (0, \infty) \to \mathbb{N} \quad \forall (\mathfrak{A}, \mathsf{L}) \in \mathscr{C} \quad \operatorname{cov}_F(\mathfrak{A}, \mathsf{L}|\varepsilon) \leqslant G(\varepsilon)$.

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Example: L. 13 and L. 15

If $d \in \mathbb{N} \setminus \{0, 1\}$, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all multipliers θ of $\mathbb{Z}^d$:

$$\operatorname{cov}\left(\left(C^*\left(\mathbb{Z}^d, \theta\right), \mathsf{L}_\theta\right)\middle|\varepsilon\right) \leqslant N.$$

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Example: Aguilar, L., 15

$$\left\{ (\mathfrak{A}, \mathsf{L}) \,\middle|\, \begin{array}{l} \mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n} \text{ with } \forall_{\mathbb{N}} n \quad \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \\ \mathfrak{A} \text{ has a faithful tracial state} \\ \forall n \in \mathbb{N} \quad d(n) \leqslant \dim \mathfrak{A}_n \leqslant D(n) < \infty \\ \forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{L}(a) = \sup_{n\in\mathbb{N}} \dim \mathfrak{A}_n \|a - \mathbb{E}_n(a)\|_{\mathfrak{A}_n} \end{array} \right\}$$

Covariance Property of Λ^*

Symmetries and metric convergence

Do symmetries or dynamics pass to the limit for the dual propinquity, under some *equicontinuity* condition ?

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Do symmetries or dynamics pass to the limit for the dual propinquity, under some *equicontinuity* condition ?

To study this question:

1. we define a distance on *proper monoids*,
2. we then define a *covariant* version of the propinquity,
3. we then prove a form of Arzéla-Ascoli theorem for the *dual propinquity*.

A proper monoid Gromov-Hausdorff distance

Let (G_1,δ_1) and (G_2,δ_2) be two proper metric monoids. A *ε-isometric isomorphism* $(\varsigma_1,\varsigma_2)$ from (G_1,δ_1) to (G_2,δ_2) is a pair of functions $\varsigma_1 : G_1 \to G_2$ and $\varsigma_2 : G_2 \to G_1$ such that for all $\{j,k\} = \{1,2\}$:

$$\forall g, g' \in G_j\left[\frac{1}{\varepsilon}\right], h \in G_k\left[\frac{1}{\varepsilon}\right]$$

$$\boxed{\left|\delta_k(\varsigma_j(g)\varsigma_j(g'),h) - \delta_j(gg',\varsigma_k(h))\right| \leqslant \varepsilon,}$$

and ς_1, ς_2 are unital.
We then set:

$$\Upsilon((G_1,\delta_1),(G_2,\delta_2)) = \boxed{\inf\left\{\varepsilon > 0 | \exists(\varsigma,\kappa)\ \varepsilon\text{-iso-iso}\right\}.}$$

Υ is a *metric up to isometric isomorphism of proper monoid*, and it dominates the pointed Gromov-Hausdorff distance.

Covariant Propinquity

We can combine Λ^* and Υ.

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- a quantum compact metric space $(\mathfrak{A}, \mathsf{L})$,
- a proper monoid (G, δ)
- a strongly continuous action of G on $\mathfrak{A}$ by Lipschitz morphisms.

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Definition (L.,18)

A *ε-covariant tunnel* $(\tau, \varsigma, \kappa)$ from $(\mathfrak{A}, \mathsf{L}, G, \delta, \alpha)$ to $(\mathfrak{B}, \mathsf{L}, H, \delta, \beta)$ is a tunnel τ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and a ε-almost isometric isomorphism $(\varsigma_1, \varsigma_2)$ from (G, δ_G) to (H, δ_H).

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Definition (L., 18)

The *ε-reach* $\rho(\tau|\varepsilon)$ of a covariant tunnel $\tau = (\mathfrak{D}, \mathsf{L}, \pi_1, \pi_2, \varsigma_1, \varsigma_2)$ from $(\mathfrak{A}_1, \mathsf{L}_1, G_1, \delta_1, \alpha_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2, G_2, \delta_2, \alpha_2)$ is:

$$\max_{\{j,k\}=\{1,2\}} \sup_{\varphi \in \mathscr{S}(\mathfrak{A}_j)} \inf_{\psi \in \mathscr{S}(\mathfrak{A}_k)} \sup_{g \in G_j\left[\frac{1}{\varepsilon}\right]} \mathsf{mk}_{\mathsf{L}}\left(\varphi \circ \alpha_j^g \circ \pi_j, \psi \circ \alpha_k \circ \alpha_k^{\varsigma_k(g)} \circ \pi_k\right).$$

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- a quantum compact metric space $(\mathfrak{A}, \mathsf{L})$,
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- a strongly continuous action of G on $\mathfrak{A}$ by Lipschitz morphisms.

Definition (L., 18)

The *covariant propinquity* $\Lambda^{\mathrm{cov}}(\mathbb{A}, \mathbb{B})$ between two Lipschitz dynamical systems $\mathbb{A}$ and $\mathbb{B}$ is:

$$\inf\left\{\varepsilon > 0 \middle| \exists \tau \text{ a } \varepsilon\text{-covariant tunnel } \mathbb{A} \to \mathbb{B} \text{ with } \max\{\chi(\tau), \rho(\tau|\varepsilon)\} \leqslant \varepsilon\right\}$$

Covariant Propinquity

We can combine Λ^* and Υ. A *Lipschitz dynamical system* $(\mathfrak{A}, \mathsf{L}, G, \delta, \alpha)$ consists of:

- a quantum compact metric space $(\mathfrak{A}, \mathsf{L})$,
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- a strongly continuous action of G on $\mathfrak{A}$ by Lipschitz morphisms.

Theorem (L., 18)

The *covariant propinquity* Λ^{cov} is a *metric* up to *equivariant* full quantum isometry, i.e it is symmetric, satisfies the triangle inequality, and:

$$\Lambda^{\mathrm{cov}}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, G, \delta_G, \alpha), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, H, \delta_H, \beta)) = 0$$

if and only if there exists a full quantum isometry $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \twoheadrightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and an isometric isomorphism $\varsigma : G \to H$ such that:

$$\forall a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) \quad \forall g \in G \quad \pi \circ \alpha^g = \beta^{\varsigma(g)} \circ \pi.$$

Theorem (L., 17, 18)

If $D : [0,\infty) \to [0,\infty)$ and:

1. $\forall n \in \mathbb{N}$, $(\mathfrak{A}_n, \mathsf{L}_n, G_n, \delta_n, \alpha_n)$ is a Lipschitz dynamical F-system,
2. $\lim_{n\to\infty} \Upsilon((G_n,\delta_n),(G,\delta)) = 0$,
3. $\lim_{n\to\infty} \Lambda^*_F((\mathfrak{A}_n,\mathsf{L}_n),(\mathfrak{A},\mathsf{L})) = 0$,
4. for all $\varepsilon > 0$ there exists $\omega > 0, N \in \mathbb{N}$ such that:

$$\forall n \geqslant N \quad \forall g,h \in G_n \quad \delta_n(g,h) < \omega \implies \sup\left\{ \left\| \alpha_n^g(a) - \alpha_n^h(a) \right\|_{\mathfrak{A}_n} \middle| \mathsf{L}_n(a) \leqslant 1 \right\} < \varepsilon,$$

then there exists an action α of G on $\mathfrak{A}$ and a $j : \mathbb{N} \to \mathbb{N}$ strictly increasing such that:

$$\lim_{n\to\infty} \Lambda^{\mathrm{cov}}_F((\mathfrak{A}_{j(n)}, \mathsf{L}_{j_n}, G_{j(n)}, \alpha_{j(n)}), (\mathfrak{A}, \mathsf{L}, G, \alpha)) = 0.$$

If G, G_n is a compact group and α_n *ergodic* $\forall_{\mathbb{N}} n$ then so is α.

Application to Closures

Theorem (L., 17, new)

Let $\mathscr{C}$ be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of $SU(2)$ by isometries. The *closure* of $\mathscr{C}$ for the dual propinquity consists only of quantum compact metric spaces over *type I* C*-algebras.

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Theorem (L.,17, alternate proof)

Let $\mathscr{F}$ be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of $\mathbb{T}^2$ by isometries. The *closure* of $\mathscr{F}$ for the dual propinquity consists of fuzzy and quantum tori.

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Theorem (L., 18, very new)

The *covariant propinquity* is *complete* on the class of Lipschitz dynamical systems with a bi-invariant metric.

Convergence of dual actions of quantum tori

Let ℓ be a *continuous length function* on $\mathbb{T}^d$. For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of $\widehat{G}$, for any $a \in C^*(\widehat{G},\sigma)$, set:

$$\boxed{\mathsf{L}_{G,\sigma}(a) = \sup\left\{ \frac{\|\alpha^g_{G,\sigma}(a) - a\|_{C^*(\widehat{G},\sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}}$$

where $\alpha_{G,\sigma}$ is the dual action of G on $C^*(\widehat{G},\sigma)$.

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Theorem (L., 18)

If $(G_n)_{n\in\mathbb{N}}$ is a sequence of closed subgroups of $\mathbb{T}^d$ converging to $\mathbb{T}^d$ for the Hausdorff distance Haus_ℓ and if σ_n are (lift of) multipliers of $\widehat{G_n}$ converging pointwise to some σ then:

$$\lim_{n\to\infty} \Lambda^{\mathrm{cov}}_F\Big(\big(C^*(\widehat{G_n},\eta), \mathsf{L}_{G_n,\eta}, G_n, \alpha_{G_n,\sigma_n}\big), \big(C^*(\mathbb{Z}^d,\sigma), \mathsf{L}_{\mathbb{T}^d,\sigma}, \mathbb{T}^d, \alpha_{\mathbb{T}^d,\sigma}\big)\Big) = 0.$$

GPS

Metrics for Vector Bundles

- Let (M,g) be a compact, connected Riemannian manifold, and let (V,h) be a vector bundle endowed with a metric.

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Thus, ΓV is a $C(M)$-Hilbert module.

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- Let ∇ be a metric connection on ΓV, i.e. $\forall X \in \Gamma TM$:

$$d_X \langle \omega, \xi \rangle = \langle \nabla_X \omega, \xi \rangle + \langle \omega, \nabla_X \xi \rangle.$$

  ∇ defines a norm on a dense subspace of ΓV:

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  Our idea is to introduce a metric on objects of the form $(\Gamma V, \langle \cdot, \cdot \rangle_{C(M)}, \mathsf{D}, C(M), \mathsf{L})$.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space*,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a *left Hilbert module* over $\mathfrak{A}$,
3. D is a *norm* on a dense subspace of $\mathscr{M}$ such that:
    1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
    2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
    3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
    4. $\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{D}(\omega), \mathsf{D}(\eta))$.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space*,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a *left Hilbert module* over $\mathfrak{A}$,
3. D is a *norm* on a dense subspace of $\mathscr{M}$ such that:
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Full quantum isometries

(θ, Θ) full quantum isometry when θ full quantum isometry between bases and $\Theta(a\xi) = \theta(a)\Theta(\xi)$, Θ linear isomorphism preserving both the norms and the D-norms.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$. Let $\mathscr{A}_\theta$ be the associated quantum torus.

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1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:
$$\alpha_{\eth}^{x,y,t} \xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$
Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

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$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0} \xi.$$
3. For Schwarz functions ξ, ω, set:
$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle u_\theta^n v_\theta^m \xi, \omega \rangle_{L^2(\mathbb{R}, \mathbb{C}^d)} u_\theta^n v_\theta^m;$$
complete space of Schwarz functions to the *Heisenberg module* $\mathscr{H}_\theta^{p,q,d}$.

The Heisenberg Modules

Fix $\theta \in [0,1)$, $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$, and $d \in q\mathbb{N}$. The canonical generators of $\mathscr{A}_\theta$ are U, V such that $UV = \exp(2i\pi\theta)VU$.

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- The candidate *D-norm* is related to the norm of the operator:

$$(x, y) \in \mathbb{R}^2 \mapsto x\nabla_1\xi + y\nabla_2\xi,$$

for $\xi \in \mathscr{S}^d$.

D-norms for Heisenberg Modules

Theorem (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set $\mathsf{D}_\theta^{p,q,d}(\xi)$ as:

$$\boxed{\sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.}$$

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

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As a note, $\mathsf{D}_\theta^{p,q,d}(\xi)$ is actually the max of operator norm of $\nabla\xi$ where ∇ is the Connes connection on $\mathscr{H}_\theta^{p,q,d}$ and the usual Hilbert module norm.

Tunnels between metrized quantum vector bundles

$$
\begin{tikzcd}
 & & (\mathscr{P}, \mathsf{D}_{\mathscr{P}}) \arrow[ddll, bend right, "\Theta_{\mathfrak{A}}"'] \arrow[ddrr, bend left, "\Theta_{\mathfrak{B}}"] & & \\
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[u, dashed] \arrow[ddll, "\theta_{\mathfrak{A}}"'] \arrow[ddrr, "\theta_{\mathfrak{B}}"] & & \\
(\mathscr{M}, \mathsf{D}_{\mathscr{M}}) & & & & (\mathscr{N}, \mathsf{D}_{\mathscr{N}}) \\
(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \arrow[u, dashed] & & & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \arrow[u, dashed]
\end{tikzcd}
$$

Figure: A modular tunnel

Definition (L., 18)

The *extent* of a modular tunnel is the extent of the underlying tunnel.

The modular propinquity

Definition (L., 16, L. 18)

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Theorem (Heisenberg Modules (L. 17))

If $(\theta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}\setminus\mathbb{Q}$ converging to some irrational number θ, and $p,q\in\mathbb{Z}\setminus\{0\}, d\in\mathbb{N}\setminus\{0\}$ then:

$$\lim_{n\to\infty}\Lambda^{\mathsf{mod}}((\mathscr{H}^{p,q,d}_{\theta_n},\mathscr{H}^{p,q,d}_{\theta}))=0.$$

Theorem (L., 17)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p, q, d fixed. If for all $\theta \in \mathbb{R}$, and $a \in \mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)} a - a\right\|_{\mathscr{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where β_θ is the dual action, and for all $\xi \in \mathscr{H}_\theta^{p,q,d}$ we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}} \xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\,\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where $\eth = \theta - p/q$, then:

$$\lim_{\vartheta\to\theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathscr{A}_\vartheta, \mathsf{L}_\vartheta\right), \left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)\right) = 0.$$

Metrical Quantum Vector Bundles

$$\begin{tikzcd}
 & & (\mathfrak{D}', \mathsf{L}'_{\mathfrak{D}}) \arrow[dddll, bend right, "\pi_{\mathfrak{A}}"'] \arrow[dddrr, bend left, "\pi_{\mathfrak{B}}"] \arrow[d, dotted] & & \\
 & & (\mathscr{P}, \mathsf{D}_{\mathscr{P}}) \arrow[ddl, bend right, "\Theta_{\mathfrak{A}}"'] \arrow[ddr, bend left, "\Theta_{\mathfrak{B}}"] & & \\
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[u, dashed] \arrow[ddl, "\theta_{\mathfrak{A}}"] \arrow[ddr, "\theta_{\mathfrak{B}}"'] & & \\
(\mathfrak{A}', \mathsf{L}'_{\mathfrak{A}}) \arrow[r, dotted] & (\mathscr{M}, \mathsf{D}_{\mathscr{M}}) & & (\mathscr{N}, \mathsf{D}_{\mathscr{N}}) & (\mathfrak{B}', \mathsf{L}'_{\mathfrak{B}}) \arrow[l, dotted] \\
 & (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \arrow[u, dashed] & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \arrow[u, dashed] &
\end{tikzcd}$$

Figure: A metrical modular tunnel

Spectral Triples

If $(\mathfrak{A}, \mathscr{H}, D)$ is a *metric* spectral triple, and if we set:

$$\mathrm{qvb}\,(\mathfrak{A}, \mathscr{H}, D) = (\mathscr{H}, \mathsf{D}_D, \mathbb{C}, 0, \mathfrak{A}, \mathsf{L})$$

where $\mathsf{D}_D(\xi) = \|\xi\|_{\mathscr{H}} + \|D\xi\|_{\mathscr{H}}$ and $\mathsf{L}(a) = |||[D, a]|||_{\mathscr{H}}$ for all $\xi \in \mathscr{H}$ and $a \in \mathfrak{A}$, then:

Theorem (L., 18)

$\Lambda^{\mathsf{mod}}(\mathrm{qvb}\,(,,), \mathrm{qvb}\,(,,))$ defines a pseudo-metric on metric spectral triples such that $(\mathfrak{A}, \mathscr{H}_{\mathfrak{A}}, D_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathscr{H}_{\mathfrak{B}}, D_{\mathfrak{B}})$ are at distance zero if, and only if there exists $U : \mathscr{H}_{\mathfrak{A}} \to \mathscr{H}_{\mathfrak{B}}$ unitary such that $U D_{\mathfrak{A}}^2 U^* = D_{\mathfrak{B}}^2$.

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What about conserving the entire information in the spectral triple?

The covariant modular propinquity

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
- *Convergence of Fuzzy Tori and Quantum Tori for the quantum Gromov-Hausdorff Propinquity: an explicit Approach*, F. Latrémolière, *Münster Journal of Mathematics* **8** (2015) 1, pp. 57–98, ArXiv: 1312.0069
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Indiana University Mathematics Journal* **66** (2017) 5, 1707–1753, Arxiv: 1501.06121
- *Quantum Ultrametrics on AF algebras*, K. Aguilar and F. Latrémolière, *Studia Math.* **231** (2015) 2, pp. 149–193, ArXiv: 1511.07114
- *Equivalence of Quantum Metrics with a common domain*, F. Latrémolière, *Journal of Math. Anal. and Appl.* **443** (2016) 2. 1179–1195, ArXiv: 1604.00755
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Indiana Univ. Math. J.* **66** (2017) 5, 1707–1753, Arxiv: 1501.06121