THE FUNDAMENTAL THEOREM OF CALCULUS

ABSTRACT. we present a proof of the fundamental theorem of calculus based upon an intuitive set of axioms for areas.

Our purpose is to compute areas defined by functions. Given an interval $[a, b]$ in $\mathbb{R}$ and a function f continuous on $[a, b]$, we would like to provide a real number $\int_a^b f$ which measures, in some sense, the area of the set of points between the graph of f, the x-axis, and the vertical lines of equation $x = a$ and $x = b$, as in Figure (1).

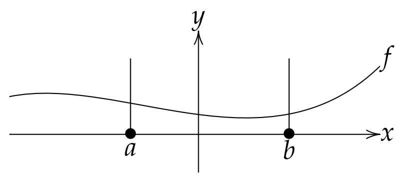

FIGURE 1. $\int_a^b f$ is a measure of the area between the graph of f and the lines of equations $y = 0$, $x = a$ and $x = b$

A proper definition of $\int$ requires a deep understanding of analysis. However, a remarkable fact is that if we give ourselves a rather minimal set of basic properties which we expect an area measurment operation to satisfy, then the solution to our problem is unique. We shall start by exploring this uniqueness property. As an additional benefit, the uniqueness result provides a mean to compute the values $\int_a^b f$ in many important cases.

We then note that our characterization of $\int$ is actually enough to suggest a construction of the integral on continuous functions using Riemann sums, and we undertake a general introduction to the existence of $\int$.

1. UNIQUENESS AND THE FUNDAMENTAL THEOREM OF CALCULUS

In this section, we *assume* that we are given an operation called integral and denoted by $\int$ which is defined on all pairs $([a, b], f)$ for any $a < b \in \mathbb{R}$ and continuous function f on $[a, b]$, return for any such datum a real number denoted by $\int_a^b f$, and satisfy the Conditions (1.1,1.2,1.3). The main reason for these conditions is that they reflect some very minimal intuition of what an operation which measures area should satisfy.

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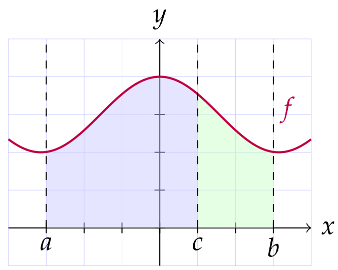
Condition 1.1. *For any $a \leqslant c \leqslant b \in \mathbb{R}$ and any continuous function f on $[a,b]$, we have:*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

This relation is known as the Chasles relation for the integral. It states that the area of the surface made as the union of two adjacent surfaces is the sum of their area.

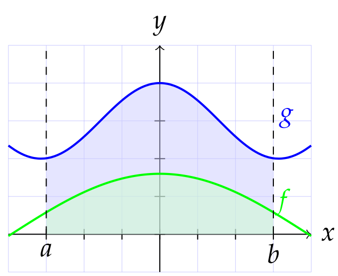


Chasles: *If $a \leqslant c \leqslant b$, then $\int_a^b f = \int_a^c f + \int_c^b f$.*

Condition 1.2. *Let $a \leqslant b \in \mathbb{R}$ and let f, g be two continuous functions on $[a,b]$. If $f \leqslant g$ on $[a,b]$, i.e. $f(x) \leqslant g(x)$ for all $x \in [a,b]$, then we have:*

$$\int_a^b f \leqslant \int_a^b g.$$

This condition is called positivity. It reflects the fact that the area under g is greater than the area under f, over the same interval $[a,b]$, if $g \geqslant f$.



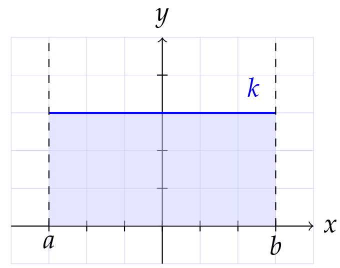
Positivity: *If $f \leqslant g$ on $[a,b]$, then $\int_a^b f \leqslant \int_a^b g$.*

Condition 1.3. *We normalize our area computations by setting, for all $a \leqslant b \in \mathbb{R}$ and $k \in \mathbb{R}$ that:*

$$\int_a^b k = k(b-a).$$

This is of course the usual area for a rectangle.

Our conditions already impose some important, simple properties on $\int$.
We start with the easiest, most obvious:



Normalization: $\int_a^b k = k(b-a)$.

Proposition 1.4. *Let $a \in \mathbb{R}$ and f be a function defined at a. Then:*

$$\int_a^a f = 0.$$

Proof. When restricted to $\{a\} = [a,a]$, the function f is constant with value $f(a)$, so by Condition (**??**), we have:

$$\int_a^a f = f(a)(a-a) = 0.$$

$\square$

Much more interesting is the following consequence of positivity and our choice of normalization:

Proposition 1.5. *Let $a \leqslant b \in \mathbb{R}$ and let f be a continuous function on $[a,b]$. Let $m, M \in \mathbb{R}$ such that for all $x \in [a,b]$ we have $m \leqslant f(x) \leqslant M$. Then:*

$$m(b-a) \leqslant \int_a^b f \leqslant M(b-a).$$

Remark 1.6. Since f is continuous on $[a,b]$, it is bounded (and reaches its bounds) by the extreme value theorem, so there always exists m, M such that $m \leqslant f \leqslant M$ on $[a,b]$.

Proof. By Condition (1.2), since $f \leqslant M$ on $[a,b]$, we conclude:

$$\int_a^b f \leqslant \int_a^b M.$$

Now by Condition (**??**), we have $\int_a^b M = M(b-a)$, hence our result. Similarly, Condition (1.2), Condition (**??**) and $m \leqslant f$ on $[a,b]$ implies $m(b-a) = \int_a^b m \leqslant \int_a^b f$. $\square$

We can extend the definition of $\int$ in a manner which preserves the Chasles relation:

Definition 1.7. Let $a \leqslant b \in \mathbb{R}$ and f be a continuous function on $[a,b]$. Then $\int_b^a f$ is the real number $-\int_a^b f$.

Proposition 1.8. *Let $a, b, c \in \mathbb{R}$ and let f be a continuous function on some interval containing a, b, c. Then:*

$$\int_a^b f + \int_b^c f + \int_c^a f = 0.$$

Proof. Assume $a \leqslant b \leqslant c$. Then by definition, $\int_c^a f = -\int_a^c f$, and thus our result is proven by Condition (1.1).

Using commutativity and associativity of the addition for real numbers, this proof also includes the cases $b \leqslant c \leqslant a$ and $c \leqslant a \leqslant b$.

Assume now $b \leqslant a \leqslant c$. Then:

$$\begin{aligned} \int_a^b f + \int_b^c f + \int_c^a f &= -\int_b^a f - \int_a^c f + \int_b^c f \\ &= -(\int_b^c f) + \int_b^c f = 0 \end{aligned}$$

as expected. Again, this proof also deals with the cases $a \leqslant c \leqslant b$ and $c \leqslant b \leqslant a$. □

The fundamental idea for this section is to introduce and study the following functions:

Definition 1.9. Let $a \leqslant b \in \mathbb{R}$ and let f be a continuous function on $[a, b]$. We define:

$$\mathscr{F}_a : x \in [a, b] \mapsto \int_a^x f.$$

Thus, $\mathscr{F}_a$ is a function which, to any element t of its domain, returns the expected area between the graph of f, the x-axis and the vertical lines $x = a$ and $x = t$.

Theorem 1.10. *Let $a \leqslant b \in \mathbb{R}$ and let f be a continuous function on $[a, b]$. The function $\mathscr{F}_a$ defined in Definition (1.9) on $[a, b]$ by:*

$$\mathscr{F}_a : x \in [a, b] \mapsto \int_a^b f$$

is continuous on $[a, b]$, differentiable on (a, b) and for all $x \in (a, b)$, we have:

$$\mathscr{F}_a'(x) = f(x).$$

Proof. Let $x \in (a, b)$ and let $h > 0$ and $x + h \in (a, b)$. Then:

$$\frac{1}{h}\left(\mathscr{F}_a(x+h) - \mathscr{F}_a(x)\right) = \frac{1}{h}\left(\int_a^{x+h} f - \int_a^x f\right) \tag{1.1}$$

$$= \frac{1}{h}\left(\int_x^{x+h} f\right) \text{ by Proposition (1.8).} \tag{1.2}$$

Now, as f is continuous on $[a, b]$, it is continuous on $[x, x + h]$, so by the Extreme Value Theorem, there exists $m_h, M_h \in [x, x + h]$ such that for all $t \in [x, x + h]$, we have:

$$f(m_h) \leqslant f(t) \leqslant f(M_h).$$

Thus, by Proposition (1.5), we have:

$$hf(m_h) \leqslant \int_x^{x+h} f \leqslant hf(M_h). \tag{1.3}$$

Together with Inequality (1.1) we obtain:

$$f(m_h) \leqslant \frac{1}{h}\left(\mathscr{F}_a(x+h) - \mathscr{F}_a(x)\right) \leqslant f(M_h).$$

Now, since $x \leqslant m_h \leqslant x+h$ and $\lim_{h\to 0^+} x = \lim_{h\to 0^+} x+h = x$, we conclude by the Squeeze Theorem that $\lim_{h\to 0^+} m_h = x$. Since f is continuous, we then conclude $\lim_{h\to 0^+} f(m_h) = f(x)$.

Similarly, we prove that $\lim_{h\to 0^+} f(M_h) = f(x)$. Hence, Ineqality (1.3) and the Squeeze theorem imply together that:

$$\lim_{h\to 0^+} \frac{1}{h}\left(\mathscr{F}_a(x+h) - \mathscr{F}_a(x)\right) = f(x).$$

The same method applies to show that $\lim_{h\to 0^-} \frac{1}{h}\left(\mathscr{F}_a(x+h) - \mathscr{F}_a(x)\right) = f(x)$, so we only sketch this part.

Let $h < 0$ be given so that $x+h \in (a,b)$. Then:

$$\begin{aligned}\frac{1}{h}\left(\mathscr{F}_a(x+h) - \mathscr{F}_a(x)\right) &= \frac{1}{h}\int_x^{x+h} f \\ &= \frac{-1}{h}\int_{x+h}^{x} f.\end{aligned}$$

By the extreme value theorem, there exist $m_h, M_h \in [x+h, x]$ such that $f(m_h) \leqslant f(t) \leqslant f(M_h)$ for all $t \in [x+h, x]$. Thus by Condition (1.2), and noting that $x+h < x$, we have:

$$-hf(m_h) \leqslant \int_{x+h}^{x} f \leqslant -hf(M_h).$$

Thus:

$$f(m_h) \leqslant \frac{1}{h}\left(\mathscr{F}_a(x+h) - \mathscr{F}_a(x)\right) \leqslant f(M_h)$$

and as before, we conclude by the Squeeze theorem that $\lim_{x\to 0^-} \frac{1}{h}\left(\mathscr{F}_a(x+h) - \mathscr{F}_a(x)\right) = f(x)$.

Since a function is continuous at a point where it is differentiable, we conclude that $\mathscr{F}_a$ is continuous on (a,b). We also note that Proposition (1.4) shows that $\mathscr{F}_a(a) = 0$. Now, if $x \in (a,b]$ then:

$$\mathscr{F}_a(x) - \mathscr{F}_a(a) = \int_a^x f.$$

Now, since f is continuous on $[a,b]$, it is bounded, i.e. there exists $m, M \in \mathbb{R}$ such that $m \leqslant f \leqslant M$ on $[a,b]$. Hence, by positivity:

$$m(x-a) \leqslant \int_a^x f \leqslant M(x-a)$$

and thus:

$$m(x-a) \leqslant \mathscr{F}_a(x) - \mathscr{F}_a(a) \leqslant M(x-a)$$

so, by the Squeeze theorem:

$$\lim_{x\to a^+} \mathscr{F}_a(x) = \mathscr{F}_a(a) = 0.$$

A similar argument shows that $\mathscr{F}_a$ is continuous on the left of b as well. This concludes our theorem. $\square$

Remark 1.11. The proof of continuity of $\mathscr{F}_a$ on $[a,b]$ could be carried out independantly of the proof of differentiability in a manner similar to the above proof for continuity on the right of a.

Remark 1.12. We observe that $\mathscr{F}_a$ has the properties required for the Mean Value Theorem to apply to it. This will prove essential later on.

2. Consequences of the Fundamental Theorem of Calculus

We shown that, assuming the existence of a real-valued operation $\int$ defined on all pairs $([a,b],f)$ of continuous functions f on an interval $[a,b]$, satisfying Conditions (1.1,1.2,**??**), then the value $\int_a^b f$ is uniquely determined by $([a,b],f)$, i.e. there is a *unique* such operation $\int$. Then we establish several very important properties this operation must have, all which prove useful in various computations.

Our work in Theorem (1.10) motivates the following definition:

Definition 2.1. Let $a \leqslant b \in \mathbb{R}$ and f a continuous function on $[a,b]$. An *antiderivative* of f on $[a,b]$ is a function F continuous on $[a,b]$, differentiable on (a,b) and such that for all $x \in (a,b)$, we have $F'(x) = f(x)$.

Remark 2.2. Theorem (1.10) proves that $\mathscr{F}_a$ is an antiderivative of f on $[a,b]$. Hence, assuming the existence of $\int$ implies that we assume that *all continuous functions on closed intervals have antiderivatives*.

The fact that antiderivatives are required to satisfy the hypothesis of the mean value theorem implies the following important consequence:

Proposition 2.3. *Let $a \leqslant b \in \mathbb{R}$ and f a continuous function on $[a,b]$. Let F be an antiderivative of f on $[a,b]$. The following hold:*

- *For any $C \in \mathbb{R}$ the function $F + C$ is also an antiderivative of f on $[a,b]$,*
- *For any antiderivative G of f on $[a,b]$ there exists $C \in \mathbb{R}$ such that $G = F + C$.*

Proof. For any constant C, the function $F + C$ is certainly continuous on $[a,b]$ and differentiable on (a,b) since F is, and $(F+C)' = F' = f$ on (a,b).

Now, let G be any antiderivative of f on $[a,b]$. Let $C = G - F$. Then C is continuous on $[a,b]$ and differentiable on (a,b), with: $C' = G' - F' = f - f = 0$ on (a,b). Hence, by the Mean Value Theorem, C is constant on $[a,b]$, and by definition $G = F + C$. □

Thus, all antiderivatives of a given continuous function on a closed interval differ from each other simply by an additive constant. This has the following remarkable consequence:

Theorem 2.4. *Let $a \leqslant b \in \mathbb{R}$ and f be a continuous function on $[a,b]$. Let F be any antiderivative of f on $[a,b]$. Then:*

$$\int_a^b f = F(b) - F(a).$$

Proof. By Theorem (1.10), $\mathscr{F}_a : x \in [a,b] \mapsto \int_a^x f$ is an antiderivative of f on $[a,b]$. Hence, by Proposition (2.3), there exists $C \in \mathbb{R}$ such that $\mathscr{F}_a + C = F$. By definition, $\mathscr{F}_a(b) = \int_a^b f$, so we get:

$$F(b) - F(a) = \mathscr{F}_a(b) + C - (\mathscr{F}_a(a) + C) = \mathscr{F}_a(b) - \mathscr{F}_a(a) = \mathscr{F}_a(b) = \int_a^b f$$

as desired. $\square$

Thus, we have the following observation:

Observation 2.5. Computing an integral may be performed by first solving the antiderivative problem: given f continuous on some interval $[a, b]$, what is one antiderivative of f on $[a, b]$?

Since antiderivatives are defined indepedantly of $\int$, we also get:

Corollary 2.6. *There is a unique real-valued operation $\int$ defined on the set of pairs $([a, b], f)$ with $[a, b]$ a closed interval of $\mathbb{R}$ and f a continuous function on $[a, b]$, and which satisfy Conditions (1.1,1.2,* **??***).*

We can now proceed to prove some very important properties of $\int$. Since integration is now related to the problem of antiderivation, we can use our knowledge of derivation to deduce mirror properties for the integral. The first trivial observation is:

Theorem 2.7. *Let $a \leqslant b \in \mathbb{R}$ and let f be a function with a continuous derivative on $[a, b]$. Then:*

$$\int_a^b f' = f(b) - f(a).$$

Proof. The function f is by construction an antiderivative of f' on $[a, b]$. $\square$

Theorem 2.8 (linearity)**.** *Let $a \leqslant b \in \mathbb{R}$, and f, g be two continuous functions on $[a, b]$. Let $\mu, \lambda \in \mathbb{R}$. Then:*

$$\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g.$$

Proof. Let F be an antiderivative of f on $[a, b]$ and G be an antiderivative of g on $[a, b]$. Then $H = \lambda F + \mu G$ is continuous on $[a, b]$, differentiable on (a, b) and $H' = \lambda F' + \mu G' = \lambda f + \mu g$. Hence H is an antiderivative of $\lambda f + \mu g$ on $[a, b]$. Hence:

$$\begin{aligned}
\int_a^b (\lambda f + \mu g) &= H(b) - H(a) \\
&= \lambda F(b) + \mu G(b) - (\lambda F(a) + \mu G(a)) \\
&= \lambda(F(b) - F(a)) + \mu(G(b) - G(a)) \\
&= \lambda \int_a^b f + \mu \int_a^b g.
\end{aligned}$$

$\square$

Thus, linearity of derivation leads to linearity of the integral. The matter is more complicated for products and compositions. We start by handling the mirror image of the Leibniz rule for derivations of products.

Theorem 2.9 (Integration by parts)**.** *Let $a \leqslant b \in \mathbb{R}$ and f, g be two functions with continuous derivatives on $[a,b]$. Then:*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Proof. We have $(fg)' = f'g + fg'$ so $fg' = (fg)' - f'g$ and thus we have:

$$\begin{aligned} \int_a^b fg' &= \int_a^b [(fg)' - f'g] \\ &= \int_a^b (fg)' - \int_a^b f'g \text{ by Theorem (2.8)} \\ &= f(b)g(b) - f(a)g(a) - \int_a^b f'g \text{ by Theorem (2.7).} \end{aligned}$$

□

The chain rule becomes the substitution rule:

Theorem 2.10 (Substitution)**.** *Let $a \leqslant b$, u be a function with a continuous derivative on $[a,b]$ and f be a continuous function on $[u(a), u(b)] \cup [u(b), u(a)]$. Then:*

$$\int_a^b ((f \circ u) \cdot u') = \int_{u(a)}^{u(b)} f.$$

Proof. Let F be an arbitrary antiderivative of f on $[a,b]$ (for instance $F = \mathscr{F}_a$). Let $g = F \circ u$ on $[a,b]$. By assumption, g is differentiable on $[a,b]$ and:

$$g'(x) = u'(x)F'(u(x)) = u'(x)f(u(x))$$

for all $x \in (a,b)$. Thus:

$$\begin{aligned} \int_a^b f \circ u \cdot u' &= \int_a^b g' \\ &= g(b) - g(a) \\ &= F(u(b)) - F(u(a)) \\ &= \int_{u(a)}^{u(b)} f. \end{aligned}$$

□

We conclude this section with two observations. First, we prove a useful inequality:

Theorem 2.11 (Median inequality)**.** *Let $a \leqslant b \in \mathbb{R}$ and f be a continuous function on $[a,b]$. Then:*

$$\left| \int_a^b f \right| \leqslant \int_a^b |f|.$$

Proof. By definition of the absolute value, $|f|$ is continuous on $[a,b]$ and for all $x \in [a,b]$ we have:

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|.$$

By Condition (1.2) and by linearity, we have:

$$-\int_a^b |f| = \int_a^b (-|f|) \leqslant \int_a^b f \leqslant \int_a^b |f|,$$

proving our inequality. $\square$

Another observation motivates the idea that integrals allow to compute averages, as found in probability theory.

Theorem 2.12. *Let $a \leqslant b \in \mathbb{R}$ and f be a continuous function on $[a, b]$. Then there exists $c \in (a, b)$ such that:*

$$\frac{1}{b-a}\int_a^b f = f(c).$$

Proof. The function $\mathscr{F}_a$ is continuous on $[a, b]$ and differentiable on (a, b) so by the mean value theorem, there exists $c \in (a, b)$ such that:

$$\mathscr{F}_a(b) - \mathscr{F}_a(a) = (b-a)\mathscr{F}_a'(c).$$

By the Fundamental Theorem of Calculus (1.10), we have $\mathscr{F}_a'(c) = f(c)$, and by definition, $\mathscr{F}_a(b) = \int_a^b f$ and $\mathscr{F}_a(a) = 0$. This proves our theorem. $\square$

The quantity $\frac{1}{b-a}\int_a^b f$ is called the *average of f*. In particular, the strong law of large number shows that $\frac{1}{b-a}\int_a^b f$ is the limit of the empirical average for any infinite sample of points taking uniformly at random in $[a, b]$.

3. Basic practices of Computations

This section serves as a transition to the practice of computation of integrals.The most important fact about integral computation is:

Observation 3.1. Integral computations of even elementary functions is not a recursive process which always lead another elementary function. Instead, it is impossible to obtain a function obtained from composing algebraic functions, exp, ln and trigonometric functions which is an antiderivative of $x \mapsto \exp(x^2)$. The absence of general algorithm for integral computations means that in practice, one has to find the right method for a given problem, if possible, based on one's judgement and intuition.

There are some algorithms to compute antiderivatives of quite complex functions, as implemented in various computer algebra systems. But these methods are often rather tedious and unapplicable for human resolution. Besides, they are never "general", i.e., as we pointed out, some functions' antiderivatives may not have a closed form using elementary functions.

To start with, we introduce two useful notations:

Notation 3.2. It is often practical to use anonymous functions and dummy variables to write integrals. If x is any free variable, we set for any continuous function f on $[a, b]$:

$$\int_a^b f(x)\,dx = \int_a^b f.$$

The choice of the dummy variable x is arbitrary, though it is preferable to choose a free symbol. Thus $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ for example.

Notation 3.3. Let f be a continuous function on $[a,b]$. All antiderivatives of f on $[a,b]$ are of the form $\int_a^b f + C$ for arbitrary real numbers C. To simplify notations and allow for anonynmous functions, if F is any antiderivative of f on $[a,b]$, we use the following

$$\int f(x)\,dx = F(x) + C(\forall x \in [a,b]),$$

where x is any free symbol. A proper definition of such a notion is that $intf$ is the set of all possible antiderivatives of f on the chosen interval, and thus $F \in \int f$ if and only if there exists $C \in \mathbb{R}$ such that for all $x \in [a,b]$ we have $F(x) = \int_a^x f + C$.. It will used informally, and could be read by stating that $xin[a,b] \mapsto \int f(x)\,dx$ is some antiderivative of f, though we do not know which one. Thus we perform computations up to an additive constant. For instance, $\varphi : x \mapsto \frac{x^2}{2}$ and $\psi : x \mapsto \frac{x^2}{2} + 1$ are both anti-derivatives of $x \mapsto x$ on $\mathbb{R}$, and $\int x\,dx$ is also such an antiderivative, though it may be φ, ψ or any other function which differs from φ by a constant. So we would write:

$$\int x\,dx = \frac{x^2}{2} + C$$

or

$$\int x\,dx = \frac{x^2}{2} + k$$

which are all valid as long as C and k are arbitrary constant. To make things even less formal, we tend to use the same constant symbol (say, C) in all computations, regardless of whether it changes as computation goes, i.e. $+C$ becomes a reminder that all our computations are up to additive constant, rather than to specify a given value for C. Last, all this is only valid when x is restricted to an interval.

With these notations, we can use our knowledge of derivation to show the following:

if $f(x) =$	**then** $\int f(x)\,dx =$
x^α (for $\alpha \in \mathbb{R}, \alpha \neq -1$)	$\frac{x^{\alpha+1}}{\alpha+1} + C$
$\frac{1}{x}$	$\ln(x) + C$
$\exp(x)$	$\exp(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C = -\arccos(x) + C$

The two last relations can be recovered using inverse trigonometric substitution, and thus are not essential, though they are sufficiently common to warrant their record in this list.

We now present how to use the basic properties of integrals to perform simple computations. It gives us an opportunity to introduce some practical notations.

Example 3.4. Linearity allows immediate computation of the integrals of polynomials:

$$\int x^5 + 6x^4 - 3x^3 + 3x^2 - \pi x + e\,dx = \frac{1}{6}x^6 + \frac{6}{5}x^5 - \frac{3}{4}x^4 + x^3 - \frac{\pi}{2}x^2 + ex + C.$$

Example 3.5 (Integration by part). Let us compute $\int_0^1 x\exp(x)\,dx$. To do so, we use integration by part. The idea is to identify two factors, one being easy to find an antiderivative for, and such that the integration by part formula leads to simpler computations. We use a differential style of notation, rather informally. Set:

$$\begin{cases} u &= x \\ dv &= \exp(x)\,dx \end{cases}$$

so that:

$$\begin{cases} du &= dx \\ v &= \exp(x). \end{cases}$$

Then:

$$\int x\exp(x)\,dx = \int u\,dv \tag{3.1}$$

$$= uv - \int v\,du \tag{3.2}$$

$$= x\exp(x) - \int \exp(x)\,dx \tag{3.3}$$

$$= x\exp(x) - \exp(x) + C. \tag{3.4}$$

Example 3.6. Integration by part can be useful to compute the antiderivative of some transcendantal functions. In $\int \ln(x)\,dx$, set $u = \ln(x)$ and $dv = dx$, so that:

$$\int \ln(x)\,dx = \int udv = uv - \int vdu = x\ln(x) - x + C.$$

Example 3.7. Substitution is also called *change of variable* formula, and the use of differential notation illustrates why. To compute, for instance, $\int \sin^3(x)\cos(x)\,dx$, we set $f(t) = t^3$ and $u(x) = \sin(x)$. Then:

$$\begin{aligned} \int \sin^3(x)\cos(x)\,dx &= \int f\circ u(x)u'(x)\,dx \\ &= \int f(u)du = \int u^3\,du \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}\sin^4(x) + C. \end{aligned}$$

4. Riemann Sums

This note assumed the existence of the integral of continuous functions on closed intervals (defined as an operation satisfying Conditions (1.1,1.2,**??**) and concluded that such an operation must be unique. The question remains, however, on whether or not this operation exists. This section explores the question of existence.

Let $a \leqslant b \in \mathbb{R}$ and f be a continuous function on $[a,b]$. Using Condition (1.1), we see that if we divide $[a,b]$ into finitely many intervals, i.e. if we pick finitely many values $a_0 = a < a_1 < a_2 < \cdots < a_{n+1} = b$, then:

$$\int_a^b f = \sum_{j=0}^{n} \int_{a_j}^{a_{j+1}} f.$$

Now, since f is assumed continuous on $[a,b]$, it is continuous on $[a_j, a_{j+1}]$ for $j = 0, \ldots, n$ and thus, by the extreme value theorem, there exist $m_j, M_j \in [a_j, a_{j+1}]$ such that $f(m_j) \leqslant f \leqslant f(M_j)$ on $[a_j, a_{j+1}]$. Thus, using Proposition (1.8), we get:

$$\sum_{j=0}^{n} f(m_j)(a_{j+1} - a_j) \leqslant \int_a^b f \leqslant \sum_{j=0}^{n} f(M_j)(a_{j+1} - a_j).$$

The sums of the left and right hand side are in fact the sum of the areas of rectangles.

Now, suppose that we keep adding new points to our subdivision $a_0 < \ldots < a_{n+1}$ so that the distances between consecutive points keep getting smaller. As we do so, our intuition dictates that the continuity of f forces the minima and maxima of f on each of $[a_j, a_{j+1}]$ get closer as well, and converge to a single value. Thus, if an integral is to exist, it would be a limit of these two sums for the proper notion of convergence. To prove this fact requires to devise such a notion of convergence, and to use a strong form of continuity called uniform continuity which always holds for continuous functions on closed intervals. These matters are left for a deeper analysis of functions and the topology of $\mathbb{R}$.

We admit that the following holds:

Theorem 4.1. *Let $a \leqslant b$ and f be a continuous function on $[a,b]$. For each $n \in \mathbb{N}$, we suppose given $n+1$ points $a = a_{0,n} < a_{1,n} < \cdots < a_{n,n} < a_{n,n+1} = b$ such that $(\max\{|a_{j+1,n} - a_{j,n}| : j = 0, \ldots, n\})_{n\in\mathbb{N}}$ is a sequence converging to 0. For each $n \in \mathbb{N}$ and $j = 0, \ldots, n$ we also suppose given $\xi_{j,n} \in [a_{j,n}, a_{j+1,n}]$. Then the sequence:*

$$\left(\sum_{j=0}^{n} f(\xi_{j,n})(a_{j+1,n} - a_{j,n}) \right)_{n\in\mathbb{N}} \tag{4.1}$$

admits a limit. Moreover, this limit is independant of the choices of subdivisions and sequences ξ, as long as they meet the above requirements.

The sums in Equation (4.1) are called *Riemann sums* for f on $[a,b]$. Their limit can be checked to satisfy Conditions (1.1,1.2,1.3). Thus, it must be the integral we were seeking.

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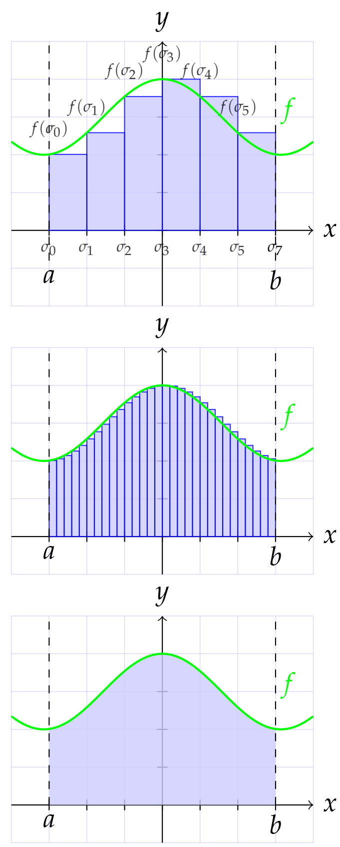

FIGURE 2. *(Riemann sums)* Approximating the area under f over $[a, b]$ using Riemann sums.