A Gromov-Hausdorff Distance for Hilbert Modules

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Pioneered by Connes (1989: spectral triples), Rieffel (1998 onward: compact quantum metric spaces).

My program

Geometry of classes of quantum metric spaces

- Study *the geometry of entire classes* of quantum metric spaces using noncommutative analogues of the *Gromov-Hausdorff distance*: the *Gromov-Hausdorff propinquity* family.
- Examples: finite dimensional approximations, perturbations of metrics, compactness theorem, ...

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- Define a *geometry for classes of modules* over quantum metric spaces: the *modular propinquity*.
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Other directions / Other talks: Development of the theory of *locally compact quantum metric spaces*, Mathematical physics.

GPS

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} : x, y \in X, x \neq y \right\}$$

for all $f \in C(X)$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{ \left| \int_X f\, d\mu - \int_X f\, d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.$$

Quasi-Leibniz Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; Kerr, 02; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *F–quasi-Leibniz quantum compact metric space* when:

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We call L an *L-seminorm*.

Ergodic actions of compact metric groups

Theorem (Rieffel, 98)

Let G be a *compact group* endowed with a *continuous length function* ℓ. Let α be an *action* of G on some *unital C*-algebra* $\mathfrak{A}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1_G\} \right\}.$$

$(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* if and only if $\left\{ a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a \right\} = \mathbb{C}1_{\mathfrak{A}}$.

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Example: Quantum tori

- $G = \mathbb{T}^d$,
- $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$ (universal for $U_j U_k = \sigma(j,k) U_{j+k}$),
- α : dual action ($\alpha^z U_j = z^j U_j$).
- Associated with a differential calculus when ℓ from invariant Riemannian metric.

Spectral triples and quantum metrics

Theorem (Rieffel, 02; Ozawa-Rieffel, 05; Christ-Rieffel, 15)

Let G be a *discrete group*, l a *length function* on G, and π be the left regular representation of $C^*_{\mathrm{red}}(G)$ on $\ell^2(G)$. For all $\xi \in \ell^2(G)$, set $D\xi : g \in G \mapsto \ell(g)\xi(g)$. If G is hyperbolic or nilpotent and l is the word-length function for some finite generator set, then $\left(C^*(G), \left|\!\left|\!\left|[D,\pi(\cdot)]\right|\!\right|\!\right|_{\mathscr{H}}\right)$ is a Leibniz quantum compact metric space.

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Theorem (Aguilar and L., 2015)

Let $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ with $\mathfrak{A}_n$ *f.d.* for all $n \in \mathbb{N}$. if $\mathfrak{A}$ is unital and has a faithful tracial state τ, and if for all $a \in \mathfrak{A}$ we set:

$$\mathsf{L}(a) = \sup\left\{ \frac{\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N} \right\}$$

where $\beta \in (0,\infty)^{\mathbb{N}}$ with $\lim_\infty \beta = 0$, and $\mathbb{E}_n : \mathfrak{A} \twoheadrightarrow \mathfrak{A}_n$ is the conditional expectation with $\tau \circ \mathbb{E}_n = \tau$, then $(\mathfrak{A}, \mathsf{L})$ *is a quasi-Leibniz quantum compact metric space*.

Other Examples

1. Quantum metrics from the standard spectral triple on quantum tori (Rieffel, 98 and 02)
2. Connes-Landi spheres (H. Li, 03)
3. Conformal deformations of quantum metric spaces from spectral triples (L., 15)
4. Curved quantum tori of Dabrowsky and Sitarz (L., 15)
5. Group C*-algebras for groups with rapid decay
6. Certain C*-crossed-products (J. Bellissard, M. Marcolli, Reihani, 10), (involves my work on locally compact quantum metric space).

Lipschitz morphisms

Definition

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from* $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ *to* $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

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Theorem (Rieffel, 00; L., 16)

Let $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ be two quasi-Leibniz quantum compact metric spaces and $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ be a unital *-morphism. Let $k \geqslant 0$. The following assertions are equivalent:

1. π is a k-Lipschitz morphism,
2. (Rieffel, 00) $\mathsf{L}_\mathfrak{B} \circ \pi \leqslant k \mathsf{L}_\mathfrak{A}$,
3. (L., 16) $\pi(\mathrm{dom}(\mathsf{L}_\mathfrak{A})) \subseteq \mathrm{dom}(\mathsf{L}_\mathfrak{B})$.

Quantum Isometries

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Theorem (McShane, 1934)

Let (Z, m) be a metric space and $X \subseteq Z$ not empty. If $f : X \to \mathbb{R}$ is a k-Lipschitz function on X then there exists a k-Lipschitz function $g : Z \to \mathbb{R}$ such that g restricts to f on X.

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$$\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a *-isomorphism such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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Quantum isometries are 1-Lipschitz morphisms.

GPS

The Gromov-Hausdorff Distance

Definition

For any two compact metric spaces (X, m_X) and (Y, m_Y), we define $\mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y)$ as:

$$\left\{ (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \middle| \begin{array}{l} (Z, \mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\}$$

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Notation

The *Hausdorff distance* on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is:

$$\inf \left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X), \iota_Y(Y)) : (Z, \mathsf{m}_Z, \iota_X, \iota_Y) \in \mathrm{Adm}(\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

A noncommutative Gromov-Hausdorff distance

$$\begin{tikzcd}
 & (Z, \mathrm{d}_Z) & \\
(X, \mathrm{d}_X) \arrow[ur, hook, "\iota_X"] & & (Y, \mathrm{d}_Y) \arrow[ul, hook', "\iota_Y"']
\end{tikzcd}$$

Figure: Gromov-Hausdorff Isometric Embeddings

$$\begin{tikzcd}
 & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"] \arrow[dr, "\pi_{\mathfrak{B}}"'] & \\
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Figure: A tunnel

The Dual Gromov-Hausdorff Propinquity

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***Definition** (The extend of a tunnel)*

The *extent* of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

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Definition (L. ,13, 14 / special case)

The *dual propinquity* $\Lambda_F^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

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Theorem (L., 13)

The dual propinquity is a *complete metric* up to *full quantum isometry*, which induces the same topology on classical compact metric spaces as the Gromov-Hausdorff distance.

Target sets of tunnels

An F-tunnel $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ gives rise to compact-set-valued function which has morphism-like properties.

1. For $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the *$l$-target set* of a is:

$$\mathfrak{t}_{\tau}(a|l) = \left\{\pi_{\mathfrak{B}}(d) \in \mathfrak{sa}(\mathfrak{B}) : \mathsf{L}(d) \leqslant l, \pi_{\mathfrak{A}}(d) = a\right\}.$$

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3. For all $a, a' \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$, $t \in \mathbb{R}$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$:

$$\mathfrak{t}_\tau(a|l) + t \cdot \mathfrak{t}_\tau\left(a' \middle| l\right) \subseteq \mathfrak{t}_\tau\left(a + ta' \middle| (1 + |t|)l\right)$$

$$\mathfrak{t}_\tau(a|l) \circ \mathfrak{t}_\tau\left(a' \middle| l\right) \subseteq \mathfrak{t}_\tau\left(a \circ a' \middle| F(\|a\|_{\mathfrak{A}} + 2l\chi(\tau), \|a'\|_{\mathfrak{A}} + 2l\chi(\tau), l, l)\right)$$

with a similar expression for the Lie product.

Bridges

Definition (A bridge)

A *bridge* $(\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$ is a unital C*-algebra $\mathfrak{D}$, two unital *-monomorphisms $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$ and $\omega \in \mathfrak{D}$ such that $\mathscr{S}(\mathfrak{D}|\omega) = \{\varphi \in \mathscr{S}(\mathfrak{D}) : \varphi(\cdot\omega) = \varphi(\omega\cdot) = \varphi\} \neq \emptyset$.

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Definition (Length of a bridge)

The length $\lambda(\gamma)$ of a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is the maximum of:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}}(\mathscr{S}(\mathfrak{A}), \pi_{\mathfrak{A}}^*(\mathscr{S}(\mathfrak{D}|\omega))), \right.$$
$$\left. \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}}(\mathscr{S}(\mathfrak{B}), \pi_{\mathfrak{B}}^*(\mathscr{S}(\mathfrak{D}|\omega)))\right\},$$

and:

$$\mathsf{Haus}_{\|\cdot\|_{\mathfrak{D}}}\left(\{a\omega \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}, \{\omega b \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{B}}(b) \leqslant 1\}\right),$$

The quantum Gromov-Hausdorff Propinquity

Theorem (L. (13))

For any bridge γ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and $\lambda > 0$ with $\lambda \geqslant \lambda(\gamma)$, the following is a tunnel of extent at most 2λ:

$$\begin{aligned}(\mathfrak{A} \oplus \mathfrak{B}, (a,b) \mapsto a, (a,b) \mapsto b,\\ (a,b) \mapsto \max\left\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda}\|\pi_{\mathfrak{A}}(a)\omega - \omega\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}\right\}.\end{aligned}$$

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Theorem (L, (13))

There exists a metric up to full quantum isometry on the class of F–quasi-Leibniz quantum compact metric spaces, called the *quantum propinquity* Λ, such that:

1. $\Lambda^* \leqslant \Lambda$ ($\leqslant \mathsf{GH}$ in classical picture),
2. $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \lambda(\gamma)$ for any bridge γ between any $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

Finite Dimensional Approximations of quantum tori

Theorem (L. (13))

If for all $n \in \mathbb{N}$, we set $\mathscr{F}_n = C^*(U_n, V_n)$ where:

$$U_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \ldots & \ldots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & & & & \\ & \rho_n & & & \\ & & \rho_n^2 & & \\ & & & \ddots & \\ & & & & \rho_n^{n-1} \end{pmatrix}$$

with $\rho_n = e^{2i\pi\frac{p_n}{n}} \neq 1$, and if $\lim_{n\to\infty} \rho_n = \rho$, then:

$$\lim_{n\to\infty} \Lambda((\mathscr{F}_n, \mathsf{L}_n), (\mathscr{A}_\rho, \mathsf{L})) = 0$$

where $\mathscr{A}_\rho = C^*(U, V)$ and U, V are universal unitaries such that $VU = \exp(2i\pi\rho)UV$, while L_n and L are L-seminorms from the dual actions, and for some *fixed* continuous length function on $\mathbb{T}^2$.

Quantum Tori and the quantum propinquity

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. For each $n \in \mathbb{N}$, let $k_n \in \overline{\mathbb{N}}^d$ and σ_n be a multiplier of $\mathbb{Z}^d_k = \mathbb{Z}^d / k_n\mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ.

Then $\lim_{n\to\infty} \Lambda\left(C^*\left(\mathbb{Z}^d, \sigma\right), C^*\left(\mathbb{Z}^d_{k_n}, \sigma_n\right)\right) = 0$, where the Lip-norms are given by the dual actions for any *fixed* length function on $\mathbb{T}^d$.

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Theorem (Aguilar and L. (15) / Informal)

The Effros-Shen AF algebras parametrized by the space of irrational real numbers (i.e. by the Baire space) are a continuous family of quasi-Leibniz quantum compact metric spaces for the quantum propinquity.

Other examples

Example: Other examples of convergence

1. Conformal perturbations of quantum metrics (L., 15)
2. Dabrowsky and Sitarz' Curved quantum tori (L., 15)
3. AF algebras as limits of their inductive sequence in a *metric* sense; UHF algebras and Effros-Shen algebras form continuous families (Aguilar and L., 15),
4. Spheres as limits of full matrix algebras (Rieffel, 15)
5. Nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces have finite dim approximations (L., 15),
6. There exists an analogue of Gromov's compactness theorem (L., 15)
7. Noncommutative solenoids form a continuous family and have approximations by quantum tori (L. and Packer, 16)
8. Closed balls for the noncommutative Lipschitz distance are totally bounded for Λ (L., 16)

GPS

1. Compact Quantum Metric Spaces
2. Convergence of quasi-Leibniz quantum compact metric space
3. Metrized quantum vector bundles
4. The modular Propinquity

Metrics for Vector Bundles

What is a mean to encode metric information on vector bundles?

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Let V be a $\mathbb{C}$-vector bundles over a compact Riemannian manifold M. A *metric* on V is given as a (smooth) section of the associated bundle of sesquilinear products on the fibers of V.

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Moreover, we always have a *metric connection* ∇, such that:

$$d_X g(\omega,\eta) = g(\nabla_X \omega,\eta) + g(\omega,\nabla_X \eta).$$

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We propose that *both* g and $|||\nabla\cdot|||$ contain metric information.

Metrized quantum vector bundles

Definition (metrized quantum vector bundle, L. (16))

A metrized quantum vector bundle $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is given by:

1. $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space,
2. $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left Hilbert module over $\mathfrak{A}$,
3. D is a norm on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
   3. $\mathsf{D}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{D}(\omega))$,
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Example: Classical picture

For a compact Riemannian manifold M and a $\mathbb{C}$-vector bundle V with a metric g and metric connection ∇, we use the inner product $\langle\omega,\eta\rangle_g = \int_M g_x(\omega_x,\eta_x)\, d\mathrm{Vol}(x)$ and $\mathsf{D}(\omega) = \max\{\|\omega\|_g, |||\nabla.\omega|||\}$.

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Example: Free modules

Given $(\mathfrak{A}, \mathsf{L})$, we set $\langle(a_1,\ldots,a_d),(b_1,\ldots,b_d)\rangle_d = \sum_{j=1}^d a_j b_j^*$ and $\mathsf{L}_d(a_1,\ldots,a_d) = \max\left\{\mathsf{L}(\Re a_j), \mathsf{L}(\Im a_j) : j \in \{1,\ldots,d\}\right\}$. Let $\mathsf{D} = \max\{\|\cdot\|_d, \mathsf{L}_d\}$. Then $(\mathfrak{A}^d, \langle,\rangle_d, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ is a metrized quantum vector bundle.

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Full quantum isometries

(θ, Θ) full quantum isometry when θ full quantum isometry between bases and $\Theta(a\xi) = \theta(a)\Theta(\xi)$, Θ linear isomorphism preserving both the norms and the D-norms.

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$.

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1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:
$$\alpha_{\eth}^{x,y,t}\xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$
Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

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Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.
2. Let $W_1, W_2 \in U(d)$ with $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$ and $W_1^n = W_2^n = 1$. We get a $\mathscr{A}_\theta = C^*(u_\theta, v_\theta)$-module with:
$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$

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$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$
3. For Schwarz functions ξ, ω, set:
$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle \varphi_{p,q,d,\eth}^{n,m} \xi, \omega \rangle_{L^2(\mathbb{R},\mathbb{C}^d)} u_\theta^n v_\theta^m;$$
complete space of Schwarz functions to the *Heisenberg module* $\mathscr{H}_\theta^{p,q,d}$.

The D-norm

Definition (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

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Theorem (L., 16)

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

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The proof rests on the compactness, for f radial, of:

$$\xi \longmapsto \alpha_\eth^f \xi = \iint_{\mathbb{R}^2} f(x,y) \alpha^{x,y,\frac{xy}{2}} \xi \, dx dy$$

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For f a linear combination of certain *Laguerre functions*, $\alpha_\eth^f$ is finite rank.

$$r \longmapsto$$

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We can fabricate an L^1-approximate unit of linear combinations of Laguerre functions using *Thangavelu* results on *Césaro sums of Laguerre expansions*.

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We then check $\alpha_{\eth}^{f}(\xi) \leqslant \mathsf{D}_\theta^{p,q,d}(\xi) \iint f(x,y)(2\pi|\eth|\|x,y\|)\,dx\,dy$.

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Many details involved which are different from Rieffel's ergodic action result (which relied on finite dimension of spectral subspaces).

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The D-norm, when restricted to Schwarz functions, is the *norm of a connection studied by Connes, Rieffel for the Yang Mills problem on the quantum 2-torus.* The connection arises from the infinitesimal rep of the Heisenberg Lie algebra form $\alpha_\eth$.

GPS

Bridges for modules

Fix $\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two metrized quantum vector bundles.

Definition (L., 16)

A *modular bridge* $\left(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right)$ is a bridge $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ and two families $(\omega_j)_{j\in J} \in \mathscr{M}_{\mathfrak{A}}$, $(\eta_j)_{j\in J} \in \mathscr{M}_{\mathfrak{B}}$ with $\mathsf{D}_{\mathfrak{A}}(\omega_j), \mathsf{D}_{\mathfrak{B}}(\eta_j) \leqslant 1$ for all $j \in J$.

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Definition (L., 16)

The *length* of a modular bridge is the maximum of the length of its basic bridge, and the sum of:

1. the maximum of $\mathsf{Haus}_k(\{\omega_j : j \in J\}, \{\omega : \mathsf{D}_{\mathfrak{A}}(\omega) \leqslant 1)$ and its counterpart in $\Omega_{\mathfrak{B}}$, where:

$$k(\omega,\xi) = \sup\left\{\|\langle\omega,\eta\rangle_{\mathfrak{A}} - \langle\xi,\eta\rangle_{\mathfrak{A}}\|_{\mathfrak{A}} : \mathsf{D}_{\mathfrak{A}}(\eta) \leqslant 1\right\},$$

2. $\max\left\{\|\pi_{\mathfrak{A}}(\langle\omega_j,\omega_k\rangle_{\mathfrak{A}})\omega - \omega\pi_{\mathfrak{B}}(\langle\eta_j,\eta_k\rangle_{\mathfrak{B}})\|_{\mathfrak{D}} : j \in J\right\}$.

The modular propinquity

Definition (L., 16)

The *modular propinquity* is the largest pseudo-metric Λ^{mod} such that $\Lambda^{\mathsf{mod}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) \leqslant \lambda\left(\gamma\right)$ for any modular γ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

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Theorem (Free modules; L., 16)

If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are quasi-Leibniz quantum compact metric space then:

$$\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant$$
$$\Lambda^{\mathsf{mod}}((\mathfrak{A}^n, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$$

where $\mathsf{D}^n_{\mathfrak{A}}(a_1, \ldots, a_n) = \max_{j=1,\ldots,n}\{\|a_j\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(\Re(a_j)), \mathsf{L}_{\mathfrak{A}}(\Im(a_j))\}$.

Theorem (L., 16)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p,q,d fixed. If for all $\theta\in\mathbb{R}$, and $a\in\mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a)=\sup\left\{\frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)}a-a\right\|_{\mathscr{A}_\theta}}{\|(x,y)\|}:(x,y)\in\mathbb{R}^2\setminus\{0\}\right\}$$

where β_θ is the dual action, and for all $\xi\in\mathscr{H}_\theta^{p,q,d}$ we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi)=\sup\left\{\frac{\left\|\alpha_{\eth}^{x,y,\frac{xy}{2}}\xi-\xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|}:(x,y)\in\mathbb{R}^2\setminus\{0\}\right\}$$

where $\eth=\theta-p/q$, then:

$$\lim_{\vartheta\to\theta}\Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d},\langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}},\mathsf{D}_\vartheta^{p,q,d},\mathscr{A}_\vartheta,\mathsf{L}_\vartheta\right),\left(\mathscr{H}_\theta^{p,q,d},\langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}},\mathsf{D}_\theta^{p,q,d},\mathscr{A}_\theta,\mathsf{L}_\theta\right)\right)=0.$$

Convergence of Heisenberg modules II