Collapse in Noncommutative Geomerty

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The Space of Noncommutative "Drums"

A. Connes introduced in 1985 a generalization of *spectral geometry* to the *noncommutative realm* by means of a structure called a *spectral triple*, i.e. a noncommutative analogue of the Dirac operators on Riemannian spin manifolds.

The Space of Noncommutative Manifolds

A *spectral triple* $\boxed{(\mathfrak{A}, \mathscr{H}, \not{D})}$ is an analogue of a first order pseudo-differential operator (of Dirac type), where

- $\mathfrak{A}$ is a unital C*-algebra, i.e. a noncommutative analogue of $C(X)$ for a compact Hausdorff space X,
- $\mathscr{H}$ is a Hilbert space on which $\mathfrak{A}$ acts, analogue to the space of sections of some spinor bundles,
- $\not{D}$ is a self-adjoint operator densely defined on $\mathscr{H}$ with compact resolvent and bounded commutator with a dense *-subalgebra of $\mathfrak{A}$.

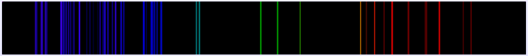

Figure: The spectrum of oxygen

The Space of Noncommutative Manifolds

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We develop a geometric framework to study the *space of (metric) spectral triples* as a natural metric space to discuss problems from mathematical physics and functional analysis.

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Past results

We define [L., 22] a *metric* up to *unitary equivalence* between *metric spectral triples* called the *spectral propinquity* Λ^{spec}, i.e.

$$\Lambda^{\mathrm{spec}}((\mathfrak{A},\mathscr{H},\mathbb{D}),(\mathfrak{B},\mathscr{J},\mathbb{P}))=0$$

iff $\exists U$ unitary from $\mathscr{H}$ onto $\mathscr{J}$ with $U\mathrm{dom}(\mathbb{D})=\mathrm{dom}(\mathbb{P})$,
$U^*\mathbb{D}U=\mathbb{P}$ and $a\in\mathfrak{A}\mapsto U^*aU$ a *-automorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.

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We proved that:

1. (L., 21) certain spectral triples from physics on fuzzy tori converge to spectral triples on noncommutative tori.
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The spectrum of spectral triples is continuous wrt spectral propinquity [L., 22], as is the continuous functional calculus in an appropriate sense.

Collapsing in Riemannian geometry

Theorem (Fukaya, 87)

If $(M_j, g_j)_{j\in\mathbb{N}}$ is a sequence of Riemannian compact manifolds converging to a compact Riemannian manifold (M, g) for the Gromov-Hausdorff distance, with uniform bounds on the sectional curvature and the diameters, then the eigenvalues of the Laplacian on (M_j, g_j) converge to the eigenvalues of the Laplacian of (M, g).

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A key step in the proof of this result is that (M_j, g_j) are like fiber bundles over (M, g) — we will discuss later the case where they are indeed such bundles, in fact, fiber bundles.

Collapsing in Riemannian geometry

Prompted by Fukaya's original work, research was carried out on finding similar results for Dirac operators. In particular, Hamman and Roos obtained interesting results along the same line.

Theorem (Abstract, Roos, 2020)

Let $(M_j, g_j)_{j\in\mathbb{N}}$ be a sequence of spin manifolds with uniform bounded curvature and diameter that converges to a lower-dimensional Riemannian manifold (B, h) in the Gromov–Hausdorff topology. Then, it happens that the spectrum of the Dirac operator converges to the spectrum of a certain first-order elliptic differential operator $\mathbb{D}_B$ on B. We give an explicit description of $\mathbb{D}_B$ and characterize the special case where $\mathbb{D}_B$ equals the Dirac operator on B.

GPS

Quantum compact Hausdorff spaces

Definition

A unital *C^*-algebra* $\mathfrak{A}$ is a unital associative algebra $\mathfrak{A}$ over $\mathbb{C}$ with a norm $\|\cdot\|_{\mathfrak{A}}$ such that:

1. $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$ is a Banach space,
2. $\forall a, b \in \mathfrak{A} \quad \|ab\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}}$,
3. there exists a conjugate linear, antimultiplicative involution $*$ on $\mathfrak{A}$ such that $\forall a \in \mathfrak{A} \quad \|a^* a\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}{}^2$.

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Theorem

A normed $*$-algebra $\mathfrak{A}$ is a C^*-algebra if, and only if $\mathfrak{A}$ is $*$-isomorphic to a closed self-adjoint algebra of bounded operators on a Hilbert space.

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- From the category of compact Hausdorff spaces: $X \mapsto C(X)$; $C(\pi : X \to Y) : f \in C(Y) \mapsto f \circ \pi \in C(X)$.

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Our work begins with a similar dual picture for *compact metric spaces*:

Founding Allegory of Noncommutative Geometry

Noncommutative geometry is the study of noncommutative generalizations of algebras of smooth functions over geometric spaces.

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Founding Allegory of Noncommutative Metric Geometry

Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\boxed{\mathsf{L}(f) = \sup\left\{\frac{|f(x) - f(y)|}{\mathsf{m}(x, y)} : x, y \in X, x \neq y\right\}}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\boxed{\mathsf{mk}_{\mathsf{L}}(\mu,\nu) = \sup\left\{ \left| \int_X f \, d\mu - \int_X f \, d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.}$$

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space* when:

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We call L an *L-seminorm*.

Metric Spectral Triples

Definition (Connes, 85)

A *spectral triple* $(\mathfrak{A}, \mathscr{H}, D\!\!\!/)$ is given by:

- a Hilbert space $\mathscr{H}$,
- a self-adjoint operator $D\!\!\!/$ defined on a sense subspace $\mathrm{dom}\,(D\!\!\!/)$ of $\mathscr{H}$, with compact resolvent,
- a unital C*-algebra $\mathfrak{A}$, *-represented on $\mathscr{H}$,

such that

$$\mathfrak{A}_{D\!\!\!/} = \{a \in \mathfrak{A} : a\,\mathrm{dom}\,(D\!\!\!/) \subseteq \mathrm{dom}\,(D\!\!\!/) \text{ and } [D\!\!\!/, a] \text{ is bounded}\}$$

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is a dense *-subalgebra of $\mathfrak{A}$.
A spectral triple is *metric* when $(\mathfrak{A}, \mathsf{L}_{\not{D}})$ is a quantum compact metric space, where

$$\forall a \in \mathfrak{A}_{\not{D}} \cap \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{L}_{\not{D}}(a) := |||[\not{D}, a]|||_{\mathscr{H}}.$$

Examples of Metric Spectral Triples

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Certain spectral triples constructed over Podleś spheres and $SU_q(2)$ are metric (Kaad, Kyed,19-22).

GPS

The Gromov-Hausdorff Distance

Te *Hausdorff distance* Haus_d between two closed subsets A_1 and A_2 of a compact metric space (X, d) is defined by

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Definition (Hausdorff, 1903; Edwards, 75; Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X,m_X) and (Y,m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X),\iota_Y(Y)) \;\middle|\; \begin{array}{l} (Z,\mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\},$$

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The *Gromov-Hausdorff distance* is a *complete metric*, up to isometry, on the class of compact metric spaces.

Quantum Isometries

A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (99), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) = \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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A *full quantum isometry* π is a *-isomorphism such that $\pi(\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})) = \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Theorem (Rieffel, 99)

If $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a quantum isometry, then $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is an isometry from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ into $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

Quantum Isometries

A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$.

Definition (Rieffel (99), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$ and

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Theorem (L., 18)

If $(\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2)$ are two *unitarily equivalent metric spectral triples*, then $(\mathfrak{A}_1, \mathsf{L}_{\not{D}_1})$ and $(\mathfrak{A}_2, \mathsf{L}_{\not{D}_2})$ are *fully quantum isometric*.

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd}[column sep=small] & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"'] \arrow[dr, "\pi_{\mathfrak{B}}"] & \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{tikzcd}$$

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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Definition (The extent of a tunnel, L. 13,14)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

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The *dual propinquity* $\Lambda^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.}$$

The Dual Gromov-Hausdorff Propinquity

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Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Theorem (L., 13)

The *dual propinquity* Λ^*, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ by:

$$\boxed{\inf\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\}}$$

is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"'] \arrow[dr, "\pi_{\mathfrak{B}}"] & \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{tikzcd}$$

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

Examples

In general, the propinquity allows one to talk about convergence of quantum compact metric spaces which are not subalgebras/quotients of each others. For instance, we have proven:

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2. $C(S^2)$ is the limit of full matrix algebras, when the metric is induced by a metric spectral triple obtained from compact Lie group actions (namely, $SU(2)$),
3. Certain fractals, constructed as unions of curves, are limits, in the propinquity, of graphs, for the geodesic distance,
4. various families of AF algebras are continuous for an appropriate quantum metric structure: the function which maps the Baire space to UHF algebras is Lipschitz for the usual metric on the Baire space and the propinquity; the function which maps an irrational number in $(0,1)$ to the Effros-Shen algebra for this number is continuous as well;

Tunnels between Spectral Triples

Let $(\mathfrak{A}_j, \mathscr{H}_j, \not{D}_j)$ be a metric spectral triple; set $\mathsf{L}_j = {|\!|\!|[\not{D}_j, \cdot]|\!|\!|}_{\mathscr{H}_j}$ and $\mathsf{D}_j(\xi) = \|\xi\|_{\mathscr{H}_j} + \left\|\not{D}_j \xi\right\|_{\mathscr{H}_j}$, $j \in \{1,2\}$.

$$\begin{tikzcd}
 & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] & \\
(\mathfrak{A}_1, \mathsf{L}_1) & & (\mathfrak{A}_2, \mathsf{L}_2)
\end{tikzcd}$$

A tunnel: $\mathsf{L}_j(a) = \inf \mathsf{L}_{\mathfrak{D}}(\theta_j^{-1}(\{a\}))$.

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$$\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[d, dashed] \arrow[ddll, bend right, "\theta_1"'] \arrow[ddrr, bend left, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[dl, bend right, "\Pi_1"'] \arrow[dr, bend left, "\Pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{H}_1, \mathsf{D}_1) & & (\mathscr{H}_2, \mathsf{D}_2) & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"]
\end{tikzcd}$$

$\mathscr{J}$ is a $\mathfrak{D}$-module, $\mathsf{D}_j(\omega) = \inf \mathsf{T}(\Pi_j^{-1}(\{\omega\}))$, T D-norm

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$$
\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dddll, "\theta_1"'] \arrow[dddrr, "\theta_2"] \arrow[d, dashed] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[ddl, "\Pi_1"'] \arrow[ddr, "\Pi_2"] \arrow[d, dotted] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{H}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{H}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathbb{C}, 0) & & (\mathbb{C}, 0) &
\end{tikzcd}
$$

$\mathscr{J}$ is a $\mathfrak{D}$-$\mathfrak{C}$-C*-corr; $(\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)$ tunnel.

Extent of Tunnels between Spectral Triples

Let $(\mathfrak{A}_j, \mathscr{H}_j, \slashed{D}_j)$ be a metric spectral triple; set $\mathsf{L}_j = \left|\left|\left|[\slashed{D}_j, \cdot]\right|\right|\right|_{\mathscr{H}_j}$ and $\mathsf{D}_j(\xi) = \|\xi\|_{\mathscr{H}_j} + \left\|\slashed{D}_j \xi\right\|_{\mathscr{H}_j}$, $j \in \{1, 2\}$.

$$\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dddll, red, "\theta_1"'] \arrow[dddrr, red, "\theta_2"] \arrow[d, dashed] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[ddl, "\Pi_1"'] \arrow[ddr, "\Pi_2"] \arrow[d, dotted] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[ddl, blue, "\pi_1"'] \arrow[ddr, blue, "\pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{H}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{H}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathbb{C}, 0) & & (\mathbb{C}, 0) &
\end{tikzcd}$$

$$\chi(\tau) = \max\left\{\chi(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2), \chi(\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)\right\}.$$

Extent of Tunnels between Spectral Triples

We can generalize this picture to tunnels between any two metrical C*-correspondences.

$$
\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dddll, red, "\theta_1"'] \arrow[d, dashed] \arrow[dddrr, red, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[ddl, "\Pi_1"'] \arrow[ddr, "\Pi_2"] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[u, dotted] \arrow[ddl, blue, "\pi_1"'] \arrow[ddr, blue, "\pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{M}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{M}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathfrak{B}, \mathsf{S}_1) & & (\mathfrak{B}_2, \mathsf{S}_2) &
\end{tikzcd}
$$

$$\chi(\tau) = \max\left\{\chi\left((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2)\right), \chi\left((\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)\right)\right\}.$$

Definition (L., 21)

If $(\mathscr{M}, \mathsf{T}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a metrical C*-correspondence, then for any $\mathbb{C}$-valued continuous linear functional $\varphi, \psi \in \mathscr{M}^*$ (where $\mathscr{M}^*$ is the $\mathbb{C}$-dual of $\mathscr{M}$), we set:

$$\mathsf{mk}_{\mathsf{T}}(\varphi, \psi) := \sup\left\{ |\varphi(\xi) - \psi(\xi)| : \xi \in \operatorname{dom}(\mathsf{T}), \mathsf{T}(\xi) \leqslant 1 \right\}.$$

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Definition (L.,22)

Let $(\mathscr{M}, \mathsf{T}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a metrical C*-correspondence. Let J be a nonempty set. For any two families $(\varphi_j)_{j\in J}, (\psi_j)_{j\in J} \in (\mathscr{M}^*)^J$ of continuous $\mathbb{C}$-linear functionals of $\mathbb{M}$, we set:

$$\mathsf{MK}_{\mathsf{T}}((\varphi_j)_{j\in J}, (\psi_j)_{j\in J}) := \sup\left\{ \mathsf{mk}_{\mathsf{T}}(\varphi_j, \psi_j) : j \in J \right\}.$$

The modular version of the Monge-Kantorovich metric

Definition (L.,21)

If $\mathbb{M} := (\mathscr{M}, \mathsf{T}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a metrical C*-correspondence, then a continuous linear functional $\varphi \in \mathscr{M}^*$ is a *pseudo-state* of $\mathbb{M}$ when there exist $\mu \in \mathscr{S}(\mathfrak{B})$ and $\omega \in \mathscr{M}$ with $\mathsf{T}(\omega) \leqslant 1$ such that φ is given by:

$$\varphi : \xi \in \mathscr{M} \longmapsto \mu\left(\langle \omega, \xi \rangle_{\mathscr{M}}\right).$$

The set of all pseudo-states of $\mathbb{M}$ is denoted by $\widetilde{\mathscr{S}}(\mathbb{M})$.

Separation and Dispersion

Definition (L.,22)

Let $\mathbb{A}$ and $\mathbb{B}$ be two metrical C*-correspondences. Let

$$\tau : \mathbb{A} \xleftarrow{(\Pi_{\mathbb{A}},\pi_{\mathbb{A}},\theta_{\mathbb{A}})} \mathbb{P} \xrightarrow{(\Pi_{\mathbb{B}},\pi_{\mathbb{B}},\theta_{\mathbb{B}})} \mathbb{B}$$

be a metrical tunnel from $\mathbb{A}$ to $\mathbb{B}$. Let T be the D-norm of the metrical C*-correspondence $\mathbb{P}$.

Let J be a nonempty set. If $A := (a_j)_{j\in J}$ is a family of operators in $\mathfrak{L}(\mathbb{A})$, and $B := (b_j)_{j\in J}$ is a family of operators in $\mathfrak{L}(\mathbb{B})$, then we define the *separation* of A and B according to τ by:

$$\begin{aligned}\mathsf{sep}\,(A,B|\tau) := \mathsf{Haus}_{\mathsf{MK}_{\mathsf{T}}}\Big(&\left\{(\varphi\circ a_j\circ\Pi_{\mathbb{A}})_{j\in J} : \varphi\in\widetilde{\mathscr{S}}(\mathbb{A})\right\}, \\ &\left\{(\psi\circ b_j\circ\Pi_{\mathbb{B}})_{j\in J} : \psi\in\widetilde{\mathscr{S}}(\mathbb{B})\right\}\Big).\end{aligned}$$

The *dispersion* $\mathsf{dis}\,(A,B|\tau)$ is $\max\{\chi(\tau), \mathsf{sep}\,(A,B|\tau)\}$.

Operational Propinquity

Definition (L.,22)

Let $\mathbb{A}$ and $\mathbb{B}$ be two metrical C*-correspondences. Let J be a nonempty set. If $A := (a_j)_{j\in J}$ is a family of operators in $\mathfrak{L}(\mathbb{A})$, and $B := (b_j)_{j\in J}$ is a family of operators in $\mathfrak{L}(\mathbb{B})$, then we define the *operational propinquity* between these families as:

$$\Lambda^{\mathsf{op}}(A, B) := \inf\left\{\mathsf{dis}\left(A, B|\tau\right) : \tau \text{ is a metrical tunnel from } \mathbb{A} \text{ to } \mathbb{B}\right\}.$$

Theorem (L.,22)

The operational propinquity Λ^{op} is a pseudo-metric on families of operators indexed by the same set. Moreover, if we fix an index set J and we fix some $j_0 \in J$, and if we set $\mathscr{F}(J)$ to be the class:

$$\mathscr{F}(J) := \Big\{ (a_j)_{j\in J} \in \mathfrak{L}(\mathbb{A})^J : \\ \mathbb{A} \text{ is a metrical C}^*\text{-correspondence, } a_{j_0} = \mathrm{id}_{\mathfrak{A}} \Big\}$$

then the restriction of Λ^{op} to $\mathscr{F}(J)$ is a metric up to the following equivalence relation. For any metrical C*-correspondences $\mathbb{A}$ and $\mathbb{B}$, and for any family $A := (a_j)_{j\in J}$ of operators in $\mathfrak{L}(\mathbb{A})$, and any family $B := (b_j)_{j\in J}$ of operators in $\mathfrak{L}(\mathbb{B})$, both families A and B being indexed by J, we have $\Lambda^{\mathsf{op}}(A,B) = 0$ if, and only if, there exists a full modular quantum isometry (Π, π, θ) from $\mathbb{A}$ to $\mathbb{B}$ such that, for all $j \in J$, we have $\Pi \circ a_j = b_j \circ \Pi$.

The Spectral Propinquity

Definition (L.,21,22)

The *spectral propinquity* between any two metric spectral triples $(\mathfrak{A}_1, \mathscr{H}_1, D_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, D_2)$ is:

$$\inf\left\{ \frac{\sqrt{2}}{2}, \varepsilon > 0 : \exists \text{ tunnel } \tau \text{ from } (\mathfrak{A}_1, \mathscr{H}_1, D_1) \text{ to } (\mathfrak{A}_2, \mathscr{H}_2, D_2) \right.$$

$$\left. \text{such that } \mathsf{dis}\left((U_1^t)_{0 \leqslant t \leqslant \frac{1}{\varepsilon}}, (U_2^t)_{0 \leqslant t \leqslant \frac{1}{\varepsilon}} \middle| \tau \right) < \varepsilon \right\},$$

The Spectral Propinquity

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$$\left. \text{such that } \mathrm{dis}\left((U_1^t)_{0 \leqslant t \leqslant \frac{1}{\varepsilon}}, (U_2^t)_{0 \leqslant t \leqslant \frac{1}{\varepsilon}} \middle| \tau \right) < \varepsilon \right\},$$

Theorem (L., 18)

The *spectral propinquity* Λ^{spec} is a *metric* on the class of spectral triples, up to unitary equivalence, i.e. $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1, \mathscr{H}_1, \mathcal{D}_1), (\mathfrak{A}_2, \mathscr{H}_2, \mathcal{D}_2)) = 0$ if, and only if there exists a *unitary* $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that $U \mathrm{dom}(\mathcal{D}_1) = \mathrm{dom}(\mathcal{D}_2)$,

$$U \mathcal{D}_1 U^* = \mathcal{D}_2 \text{ and } \mathrm{Ad}_U \text{ *-isomorphism from } \mathfrak{A}_1 \text{ to } \mathfrak{A}_2.$$

Spectrum Continuity

Theorem (

spectrum-thm If the sequence $(\mathfrak{A}_n, \mathscr{H}_n, \not{D}_n)_{n\in\mathbb{N}}$ of metric spectral triples converges to the metric spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \not{D}_\infty)$ for the spectral propinquity, then

$$\mathrm{Sp}\left(\not{D}_\infty\right) = \left\{\lambda \in \mathbb{R} : \exists (\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \quad \forall n \in \mathbb{N} \quad \lambda_n \in \mathrm{Sp}\left(\not{D}_n\right) \text{ and } \lambda = \lim_{n\to\infty} \lambda_n\right.$$

Theorem

If $(\mathfrak{A}_n, \mathscr{H}_n, \not{D}_n)_{n\in\mathbb{N}}$ *is a sequence of metric spectral triples converging to* $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \not{D}_\infty)$*, then*

$$\forall f \in C_0(\mathbb{R}) \qquad \lim_{n\to\infty} \mathsf{sep}\left(((U_n^t)_{t\in C_n}, f(\not{D}_n)), ((U_\infty^t)_{t\in C_n}, f(\not{D}_\infty)) \middle| \tau_n\right) = 0,$$

with $U_n^t := \exp(i t \not{D}_n)$.
In particular, $\lim_{n\to\infty} \Lambda^{\mathsf{op}}(f(\not{D}_n), f(\not{D}_\infty)) = 0$ *for all* $f \in C_0(\mathbb{R})$*, in the sense of Definition (??).*

GPS

The Setup for Collapse to a Base along a Vertical Direction

Let $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ be a metric spectral triple. We want to formalize the idea of splitting it into a "vertical" and a "horizontal" component.

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Now, assume $\mathbb{D} = \mathbb{D}_h + \mathbb{D}_v$, both self-adjoint operators, and let p be the projection on $\ker \mathbb{D}_v$.

We make several assumptions:

1. $[\mathbb{D}_v, b] = 0$ for all $b \in \mathfrak{B}$,
2. $(\mathfrak{B}, \ker \mathbb{D}_v, p\mathbb{D}_v p)$ is a metric spectral triple over $\mathfrak{B}$,
3. $[p, \mathbb{D}_h] = 0$,
4. there exists a conditional expectation $\mathbb{E} : \mathfrak{A} \to \mathfrak{B}$ and $k > 0$ such that for all $a \in \mathfrak{A}_{\mathbb{D}}$,
$$\|a - \mathbb{E}(a)\|_{\mathfrak{A}} \leqslant k |||[\mathbb{D}_v, a]|||_{\mathscr{H}}$$
and
$$\left|\left|\left| p[\mathbb{D}_h, \mathbb{E}(a)]p \right|\right|\right|_{\mathscr{H}} = |||[\mathbb{D}_h, \mathbb{E}(a)]|||_{\mathscr{H}} \leqslant |||[\mathbb{D}_h, a]|||_{\mathscr{H}}.$$
5. $\left\{|||[\mathbb{D}_v, a]|||_{\mathscr{H}}, |||[\mathbb{D}_h, a]|||_{\mathscr{H}}\right\} \leqslant |||[\mathbb{D}, a]|||_{\mathscr{H}}$ for all $a \in \mathfrak{A}_{\mathbb{D}}$.

A collapse theorem

Theorem (L., 24)

We assume given the above. Assume moreover that 0 is an isolated point in $\mathrm{Sp}(\mathbb{D}_v)$.

For all $\varepsilon \in (0,1)$, the triple $(\mathfrak{A}, \mathscr{H}, \mathbb{D}_h + \frac{1}{\varepsilon}\mathbb{D}_v)$ is a metric spectral triple, and moreover:

$$\lim_{\varepsilon \to 0} \Lambda^{\mathrm{spec}}((\mathfrak{A}, \mathscr{H}, \mathbb{D}_h + \frac{1}{\varepsilon}\mathbb{D}_v), (\mathfrak{B}, \ker \mathbb{D}_v, p\mathbb{D}_h p)) = 0.$$

Collapsing to a point

What are the spectral triples over a point?

$$\boxed{(\mathbb{C}, \mathbb{C}^n, \not{D})}$$

with

1. n *any* strictly positive natural number,
2. $\not{D}$ is *any* self-adjoint operator,
3. $z \in \mathbb{C}$ acts as zI_n.

Theorem (L.,24)

If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a metric spectral triple, then so is $(\mathfrak{A}, \mathscr{H}, \frac{1}{\varepsilon}\not{D})$, and moreover, if $0 \in \mathrm{Sp}(\not{D})$, then

$$\lim_{\varepsilon \to 0} \Lambda^{\mathrm{spec}}\left((\mathfrak{A}, \mathscr{H}, \frac{1}{\varepsilon}\not{D}), (\mathbb{C}, \ker \not{D}, 0)\right) = 0.$$

Collapsing a product

Theorem (L,. 24)

Let $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ be a metric even spectral triple, with grading γ, and $(\mathfrak{B}, \mathscr{H}_{\mathfrak{B}}, \mathbb{D}_{\mathfrak{B}})$ be a metric spectral triple. Set

$$\mathbb{D}_{\varepsilon} := \mathbb{D}_{\mathfrak{A}} \otimes 1_{\mathscr{H}_{\mathfrak{B}}} + \gamma \otimes \frac{1}{\varepsilon} \mathbb{D}_{\mathfrak{B}}.$$

Then $(\mathfrak{A} \otimes \mathfrak{B}, \mathscr{H} \otimes \mathscr{H}_{\mathfrak{B}}, \mathbb{D}_{\varepsilon})$ is a metric spectral triple. If $0 \in \mathrm{Sp}(\mathbb{D}_{\mathfrak{B}})$, then

$$\lim_{\varepsilon \to 0} \Lambda^{\mathrm{spec}}((\mathfrak{A} \otimes \mathfrak{B}, \mathscr{H} \otimes \mathscr{H}_{\mathfrak{B}}, \mathbb{D}_{\varepsilon}), (\mathfrak{A}, \mathscr{H} \otimes \ker \mathbb{D}_{\mathfrak{B}}, \mathbb{D}_{\mathfrak{A}} \otimes 1_{\mathscr{H}_{\mathfrak{B}}})) = 0.$$

Collapsing a product

What if $(\mathfrak{A}, \mathbb{D}, \mathscr{H})$ is odd, then what do we do? We make it even! Choose γ_1, γ_2 anticommuting self-adjoint unitaries,

$$\forall j,k \in \{1,2\} \quad \gamma_j\gamma_k + \gamma_k\gamma_j = 2\mathrm{Kronecker}_j^k.$$

The spectral triple $(\mathfrak{A}, \mathscr{H} \otimes \mathbb{C}^2, \mathbb{D} \otimes \gamma_1)$ is an even spectral triple, with grading $1_{\mathscr{H}_{\mathfrak{A}}} \otimes \gamma_2$. We thus end up with the product:

$$\mathbb{D}_\varepsilon := \mathbb{D}_{\mathfrak{A}} \otimes 1_{\mathscr{H}_{\mathfrak{B}}} \otimes \gamma_1 + 1_{\mathscr{H}} \otimes \mathbb{D}_{\mathfrak{B}} \otimes \gamma_2,$$

and if $0 \in \mathrm{Sp}(\mathbb{D}_{\mathfrak{B}})$, then

$$\lim_{\varepsilon\to 0} \Lambda^{\mathrm{spec}}((\mathfrak{A}\otimes\mathfrak{B}, \mathscr{H} \otimes \mathscr{H}_{\mathfrak{B}} \otimes \mathbb{C}^2, \mathbb{D}_\varepsilon), (\mathfrak{A}, \mathscr{H} \otimes \mathbb{C}^2 \otimes \ker \mathbb{D}_{\mathfrak{B}}, \mathbb{D}_{\mathfrak{A}} \otimes 1_{\mathscr{H}_{\mathfrak{B}}} \otimes \gamma_2) = 0.$$

Noncommutative Principal Bundles : The Ingredients

Let α be a strongly continuous action of a compact Lie group G on a unital C*-algebra $\mathfrak{A}$, and

- We denote by

$$\mathfrak{B} := \{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\},$$

  the fixed point C*-subalgebra of α;
- We now assume given a metric spectral triple $(\mathfrak{B}, \mathscr{H}_{\mathfrak{B}}, \not{D}_{\mathfrak{B}})$, where we identify $\mathfrak{B}$ with its faithful image as an algebra of operators over $\mathscr{H}$;
- We also endow G with a left-invariant Riemannian metric, by choosing some inner product $\langle \cdot, \cdot \rangle_G$ on the Lie algebra $\mathfrak{g}$ of G.

Noncommutative Principal Bundles : Isotropy

We denote by $\widehat{G}$ the set of unitary-equivalence classes of irreducible representations of G Since G is compact, all its irreducible representations are finite dimensional, and, by abuse of notation again, we write $\dim\sigma$ for the dimension of the space V_σ on which σ acts (which is obviously an invariant for the class of all representations unitary equivalent to σ).

If $\sigma \in \widehat{G}$ with character $\chi_\sigma : g \in G \mapsto \mathrm{tr}(\sigma(g))$. The *isotopic subspace*, of α associated with σ is then defined as:

$$\mathfrak{A}(\sigma) := \left\{ a \in \mathfrak{A} : a = \int_G \alpha^{\chi_\sigma(g)}(a)\, dg \right\}.$$

The space $\mathfrak{A}(\sigma)$ is a Hilbert right $\mathfrak{B}$-module, with $\mathfrak{B}$-valued inner product

$$\forall a, b \in \mathfrak{A}(\sigma) \quad \langle a, b\rangle_{\mathfrak{B}} := \mathbb{E}(a^* b).$$

Moreover, $\mathfrak{A}$ is the closure of the sum $\oplus_{\sigma\in\widehat{G}}\mathfrak{A}(\sigma)$.

Free Actions

To discuss free actions. we introduce *mutliplicity spaces*, and defined as a fixed point space as follows:

$$\Gamma_{\mathfrak{A}}(\sigma) := \left\{ x \in \mathfrak{A} \otimes V_\sigma : \forall g \in G \quad \alpha^g \otimes \overline{\sigma}^g(x) = x \right\}.$$

The relationship between the isotopic space and the multiplicity space is given by the existence of a Hilbert right $\mathfrak{B}$-module isomorphism:

$$\Phi_\sigma : \sum a \otimes v \otimes w \Gamma_{\mathfrak{A}}(\sigma) \otimes V_\sigma \longrightarrow \sum \langle v, w \rangle a \in \mathfrak{A}(\sigma).$$

We henceforth assume that α is *free*: we therefore assume that, for all $\sigma \in \widehat{G}$, we have $1_{\mathfrak{B}} \in \langle \Gamma_{\mathfrak{A}}(\sigma), \Gamma_{\mathfrak{A}}(\sigma) \rangle_{\mathfrak{B}}$, following *Schwieger, Wagner, 17,21*.

The key here is that $\Gamma_{\mathfrak{A}}(\sigma)$ a *finitely generated projective* module over $\mathfrak{B}$, i.e. it is isomorphic to $P(\sigma)(\mathfrak{B} \otimes \mathscr{H}_\sigma)$ for a projection $P(\sigma)$. Using the isomorphism Φ_σ, we then get that

$$\mathfrak{A}(\sigma) = \Phi_\sigma(P(\sigma)(\mathfrak{B} \otimes \mathscr{H}_\sigma) \otimes V_\sigma).$$

Furthermore, it is proven, with careful investigation of the above

Free Actions

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In other words, $\mathfrak{A}(\sigma)$ is also isomorphic, as a $\mathfrak{B}$-module, to a finitely generated projdctive $\mathfrak{B}$-module. Moreover, we can give a useful description of $\mathfrak{A}(\sigma)$, as the closure in $\mathfrak{A}$ of the linear span of elements $a_\sigma(b, v, w) := \Phi_\sigma(P(\sigma)(b \otimes v) \otimes w)$, for all $b \in \mathfrak{B}$, $v \in \mathscr{H}_\sigma$ and $w \in V_\sigma$.

Furthermore, it is proven, with careful investigation of the above

Inducing the Spectral Triple

Now, let $(\mathfrak{B}, \mathscr{H}_{\mathfrak{B}}, \mathbb{D}_{\mathfrak{B}})$ be a metric spectral triple.

We induced the $*$-representation of $\mathfrak{B}$ to a $*$-representation of $\mathfrak{A}$ as follows. We define $\mathscr{H}_G := \overline{\oplus}_{\sigma \in \widehat{G}} \mathscr{H}_\sigma \otimes V_{\overline{\sigma}}$. We have a projection $p(\sigma)$ on $\mathscr{H}_{\mathfrak{B}} \otimes \mathscr{H}_\sigma$

We define $p := \oplus_{\sigma \in \widehat{G}} p(\sigma) \otimes \mathrm{id}_{V_{\overline{\sigma}}}$ as acting on the Hilbert sum $\oplus_{\sigma \in \widehat{G}} \mathscr{H}_{\mathfrak{B}} \otimes \mathscr{H}_\sigma \otimes V_{\overline{\sigma}}$. We then define our Hilbert space for our spectral triple over $\mathfrak{A}$ as:

$$\mathscr{H}_p := p\left(\mathscr{H}_{\mathfrak{B}} \otimes \mathscr{H}_G\right).$$

We can extend the map $\sigma \in \widehat{G} \to S(\sigma)$ to a map from the class of all representations of G in a functorial way, by setting, for any unitary representation σ of $\mathfrak{G}$, with decomposition $\sigma = \oplus_{j=1}^d \sigma_j$ in irreducible representations $\sigma_1, \ldots, \sigma_d \in \widehat{G}$:

$$S(\sigma) := \oplus_{j=1}^d S(\sigma_j).$$

Cont'd

With this in mind, we introduce, for all $\sigma \in \widehat{G}$:

$$\gamma_\sigma : b \in \mathfrak{B} \mapsto S(\sigma)(b \otimes 1_{V_\sigma})S(\sigma)^* \in \mathfrak{B} \otimes \mathfrak{L}(\mathscr{H}_\sigma),$$

and, for all $\sigma, \tau \in \widehat{G}$:

$$\omega(\sigma,\tau) = S(\sigma \otimes \tau)S(\sigma)^* S(\tau)^* \in \mathfrak{B} \otimes \mathfrak{L}(\mathscr{H}_\sigma \otimes \mathscr{H}_\tau, \mathscr{H}_{\sigma\otimes\tau}).$$

With all these notation, we build a *-representation π of $\mathfrak{A}$ on $\mathscr{H}_p$, as follows. If we pick $a_\sigma(b, v, w)$, for $\sigma \in \widehat{G}$, $b \in \mathfrak{B}, v \in \mathscr{H}_\sigma$ and $w \in V_\sigma$, and if we pick a vector $\psi_\tau(\xi,\omega,\eta) = p(\tau)(\xi \otimes \omega) \otimes \eta$, for $\tau \in \widehat{G}, \xi \in \mathscr{H}_{\mathfrak{B}}, \omega \in \mathscr{H}_\sigma$ and $\eta \in V_{\overline{\sigma}}$, then $\pi(a_\sigma(b, v, w))\psi_\sigma(\xi, \ldots)$ is defined by:

$$\pi(a_\sigma(b, v, w))\psi_\tau(\xi,\omega,\eta) := \psi_{\sigma\otimes\tau}(\omega(\tau,\sigma)\gamma_\tau(b)_{13}(\xi \otimes v \otimes \omega \otimes w \otimes \eta)).$$

Since the linear span of $\{a_\sigma(b, v, w) : \sigma \in \widehat{G}, b \in \mathfrak{B}, v \in \mathscr{H}_\sigma, w \in V_\sigma\}$ is dense in $\mathfrak{A}$, and the linear span of $\{\psi_\sigma(\xi, v, w) : \sigma \in \widehat{G}, \xi \in \mathscr{H}_{\mathfrak{B}},\}$ is dense in $\mathscr{H}_p$, it is a technical matter to check that these formulas indeed define a *-representation π of $\mathfrak{A}$ on $\mathscr{H}_p$.

The action

Now, if $g \in G$, then we define

$$u^g \psi_\sigma(\xi \otimes v \otimes w) := \psi_\sigma(\xi \otimes v \otimes \sigma^g w).$$

Thus defined, u extends to a unitary representation of G on $\mathscr{H}_p$. Owing to properties of the isometries $S(\sigma)$ [**?**, Lemma 3.3][**?**, Lemma 3.1], It is shown in [**?**, Theorem] that (π, u) is a indeed the sought-after covariant representation of $(\mathfrak{A}, G, \alpha)$. Moreover, by construction, the fixed point subspace of u is exactly $\mathscr{H}_{\mathfrak{B}} \otimes \mathscr{H}_1 \otimes V_1 = \mathscr{H}_{\mathfrak{B}}$.

Inducing Spectral Triples

We now have the tools necessary to describe the spectral triple over $\mathfrak{A}$ with which we will work. We now set, on the subapce $\mathrm{dom}\,(\mathbb{D}_h) := \oplus_{\sigma\in\widehat{G}}\mathrm{dom}\,(\mathbb{D}_{\mathfrak{B}}) \otimes \mathscr{H}_\sigma \otimes V_{\overline{\sigma}}$:

$$\mathbb{D}_h := p(\sigma)(\mathbb{D}_{\mathfrak{B}} \otimes \mathrm{id}_{\mathscr{H}_\sigma})\, p(\sigma) \otimes 1_{V_{\overline{\sigma}}},$$

and without further mention, we also write $\mathbb{D}_h$ for the closure of the above operator, which is indeed essentially self-adjoint. This operator will be our horizontal component. The operator $\mathbb{D}_h$ commutes with the action u, namely for all $g \in G$, we have $u^g\mathrm{dom}\,(\mathbb{D}_h) \subseteq \mathrm{dom}\,(\mathbb{D}_h)$ and

$$u^g\,\mathbb{D}_h = \mathbb{D}_h u^g.$$

Under natural assumptions, the dense subspace

$$\mathfrak{A}_0 := \left\{a_\sigma(b, v, w) : \sigma \in \widehat{G}, b \in \mathfrak{B}_0, v \in \mathscr{H}_\sigma, w \in V_\sigma\right\}$$

indeed has bounded commutator with $\mathbb{D}_h$.

Vertical Component

The vertical component $\mathbb{D}_v$ of our spectral triple on $\mathfrak{A}$ is constructed following Rieffel's construction. For all ξ in the *algebraic sum* $\oplus_{\sigma \in \widehat{G}} \mathscr{H}_{\mathfrak{B}} \otimes \mathscr{H}_\sigma \otimes V_\sigma$ (which is dense in $\mathscr{H}_{\mathfrak{B}} \otimes \mathscr{H}_G$ by definition), the following limit is well-defined:

$$\partial_X \xi := \lim_{t \to 0} \frac{1}{t} \left(\alpha^{\exp(tX)} \xi - \xi \right).$$

Thus, we fix an orthonormal basis $e_1, \ldots, e_d$ of the Lie algebra $\mathfrak{g}$ of G, for $\langle \cdot, \cdot \rangle_G$, and we write $\partial_j := \partial_{X_j}$ for each $j \in \{1, \ldots, d\}$. We also choose $d+1$ anticommuting, self-adjoint unitaries $\gamma_0, \ldots, \gamma_d$. We then set $\mathbb{D}_v$ to be the closure of the essentially self-adjoint operator

$$\sum_{j=1}^{d} \partial_j \otimes \gamma_j \text{ on } \oplus_{\sigma \in \widehat{G}} \mathscr{H}_{\mathfrak{B}} \otimes \mathscr{H}_\sigma \otimes V_\sigma$$

Now we obtain a spectral triple $(\mathfrak{A}, \mathscr{H}_p, \mathbb{D})$ over $\mathfrak{A}$ by setting

$$\mathbb{D} := \mathbb{D}_h \otimes \gamma_0 + \mathbb{D}_v.$$

Collapsing the fibers

Now, the construction of $\mathbb{D}_v$, hence $\mathbb{D}$, depends on our choice of a metric over G. If we replace $\langle\cdot,\cdot\rangle_G$ in our construction by $\varepsilon\langle\cdot,\cdot\rangle_G$, then let us denote by $\mathbb{D}_v^\varepsilon$ the vertical operator constructed above, and $\mathbb{D}^\varepsilon := \mathbb{D}_h \otimes \gamma_0 + \mathbb{D}_v^\varepsilon$. A direct compution shows that $\mathbb{D}_v^\varepsilon = \frac{1}{\varepsilon}\mathbb{D}_v$. Of course, “collapsing” the fibers means taking the metric along the fiber to 0, i.e. ε to 0.

We are now ready to state our main result for this section:

Theorem (U)

nder the assumption of this section, if we set $\mathbb{D} := \mathbb{D}_h + \mathbb{D}_v^\varepsilon$, then:

$$\lim_{\varepsilon\to 0} \Lambda^{\mathrm{spec}}((\mathfrak{A}, \mathscr{H}_p, \mathbb{D}_\varepsilon), (\mathfrak{B}, \ker \mathbb{D}_v, p\mathbb{D}_h p)) = 0.$$

Applications

- Spectral triples on C^*-crossed products for equicontinuous actions collapse to the spectral triple on the base C^*-algebra,
- Spectral triples on various bundles over quantum tori collapse to quantum tori
- Compact Lie group G collapse to ${}^{G}\!/_{H}$.

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Indiana Univ. Math. J.* **66** (2017) 5, 1707–1753, Arxiv: 1501.06121
- *The modular Gromov-Hausdorff propinquity*, F. Latrémolière, Dissertationes Math. 544 (2019), 70 pp. 46L89 (46L30 58B34)
- *The Gromov-Hausdorff propinquity for metric Spectral Triples*, F. Latrémolière, *Adv. Math.* 404 (2022), paper no. 108393, arXiv:1811.04534
- *Metric approximations of spectral triples on the Sierpiński gasket and other fractal curves*, T. Landry, M. Lapidus, F. Latrémolière, *Adv. Math.* 385 (2021), Paper No. 107771, 43 pp, arXiv:2010.06921.
- *Convergence of Spectral Triples on Fuzzy Tori to Spectral Triples on Quantum Tori*, F. Latrémolière, *Comm. Math. Phys.* **388** (2021) 2, 1049–1128, arXiv:2102.03729.