A Gromov-Hausdorff distance for C-dynamical systems*

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4. The actions will be by *Lipschitz morphisms*.

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\boxed{\mathsf{L}(f) = \sup\left\{\frac{|f(x) - f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y\right\}}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\mathsf{mk}_{\mathsf{L}}(\mu, \nu) = \sup\left\{ \left| \int_X f\, d\mu - \int_X f\, d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.$$

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The Gelfand map $x \in (X, \mathsf{m}) \longmapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

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We call L an *L-seminorm*.

Quantum Tori as Compact Quantum Metric Spaces

Theorem (Rieffel, 98)

If α is a strongly continuous *action* of a *compact group G*, if ℓ is a continuous *length* function over G, and if for all $a \in \mathfrak{A}$ we set:

$$\boxed{\mathsf{L}(a) = \sup\left\{\frac{\|\alpha^g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\}\right\},}$$

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Example

- *Quantum tori* are quantum compact metric spaces with $\mathfrak{A} = C^*(\mathbb{Z}^d, \sigma)$, $G = \mathbb{T}^d$ and ℓ any continuous length on $\mathbb{T}^d$;
- *Noncommutative solenoids* $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]^2, \sigma\right)$ with $G = \mathscr{S}_p^2$ (L., Packer, 16).

The Dual Gromov-Hausdorff Propinquity

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Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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Definition *(The extent of a tunnel, L. 13,14)*

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\boxed{\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{E}}^{*}(\mathscr{S}(\mathfrak{E}))\right) : \mathfrak{E} \in \{\mathfrak{A}, \mathfrak{B}\}\right\}.}$$

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The *dual propinquity* $\Lambda_F^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any } F\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.}$$

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Theorem (L., 13)

The *dual propinquity* Λ^*_F, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ by:

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is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

Quantum and Fuzzy Tori

Example:

Let ℓ be a *continuous length function* on $\mathbb{T}^d$. For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of $\widehat{G}$, for any $a \in C^*(\widehat{G},\sigma)$, set:

$$\boxed{\mathsf{L}_{G,\sigma}(a) = \sup\left\{ \frac{\|\alpha^g(a) - a\|_{C^*(\widehat{G},\sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}}$$

where α is the dual action of G on $C^*(\widehat{G},\sigma)$.

Rieffel showed in 1998 that $\left(C^*(\widehat{G},\sigma), \mathsf{L}_{G,\sigma}\right)$ is a Leibniz quantum compact metric space.

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Example: L., 13

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where α is the dual action of G on $C^*(\widehat{G},\sigma)$.
If $(G_n)_{n\in\mathbb{N}}$ is a *sequence of closed subgroups* of $\mathbb{T}^d$ *converging* to $\mathbb{T}^d$ for the *Hausdorff distance* Haus_ℓ, and if $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of multipliers of $\mathbb{Z}^d$ converging pointwise to some σ, with $\sigma_n(g) = 1$ if g is the coset of 0 for $\widehat{G_n}$, then:

$$\lim_{n\to\infty} \Lambda^*((C^*(\widehat{G_n},\sigma_n), \mathsf{L}_{\widehat{G_n},\sigma_n}), (C^*(\mathbb{Z}^d,\sigma), \mathsf{L}_{\mathbb{Z}^d,\sigma})) = 0.$$

Bridges

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An idea is to embedd $\mathfrak{A}$ and $\mathfrak{B}$ into a unital C*-algebra $\mathfrak{D}$, pick $\mathfrak{D}$ and compute:

$$\varsigma(\mathfrak{D}, \omega) = \max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}}(\mathscr{S}(\mathfrak{A}), \{\varphi \in \mathscr{S}(\mathfrak{A}) : \varphi(\omega\cdot) = \varphi(\cdot\omega) = \varphi\{\right.$$

Covariance Property of Λ^*

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To study this question:

1. we define a distance on *proper monoids*,
2. we then define a *covariant* version of the propinquity,
3. we then prove a form of Arzéla-Ascoli theorem for the *dual propinquity*.

Lipschitz morphisms

Theorem-Definition (Lipschitz Morphisms)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ such that any of the following equivalent statement holds:

1. $(\exists k)$ $\varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi$ is a *k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$*,
2. (Rieffel, 00) $(\exists k)$ $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant k \mathsf{L}_{\mathfrak{A}}$,
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Lipschitz morphisms are arrows for a category over quasi-Leibniz quantum compact metric spaces.

A distance on morphisms

Definition (L., 16)

Let $(\mathfrak{A}, \mathsf{L})$ be a quantum compact metric space and $\mathfrak{B}$ be a unital C*-algebra. For any two linear maps $\varphi, \psi : \mathfrak{A} \to \mathfrak{B}$ preserving the units and intertwining the adjoint operations, let:

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The topology induced by mk_{L} on the space of *-morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is the topology of pointwise convergence, where a net $(\alpha_j)_{j \in J}$ converges to some α when:

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Moreover, a subset Ξ of *-morphisms is totally bounded for mk_{L} if there exists $C > 0$ such that $\mathsf{L} \leqslant C \mathsf{L} \circ \alpha$ for all $\alpha \in \Xi$.

A proper monoid Gromov-Hausdorff distance

Let (G_1,δ_1) and (G_2,δ_2) be two proper metric monoids. A *ε-isometric isomorphism* $(\varsigma_1,\varsigma_2)$ from (G_1,δ_1) to (G_2,δ_2) is a pair of functions $\varsigma_1 : G_1 \to G_2$ and $\varsigma_2 : G_2 \to G_1$ such that for all $\{j,k\} = \{1,2\}$:

$$\forall g, g' \in G_j\left[\frac{1}{\varepsilon}\right], h \in G_k\left[\frac{1}{\varepsilon}\right]$$

$$\boxed{\left|\delta_k(\varsigma_j(g)\varsigma_j(g'),h) - \delta_j(gg',\varsigma_k(h))\right| \leqslant \varepsilon,}$$

and ς_1, ς_2 are unital.

We then set:

$$\Upsilon((G_1,\delta_1),(G_2,\delta_2)) = \boxed{\min\left\{\frac{\sqrt{2}}{2}, \inf\left\{\varepsilon > 0 \middle| \exists(\varsigma,\kappa)\ \varepsilon\text{-iso-iso}\right\}\right\}}.$$

Υ is a *metric up to isometric isomorphism of proper monoid*, and it dominates the pointed Gromov-Hausdorff distance.

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Definition (L.,18)

A *ε-covariant tunnel* $(\tau, \varsigma, \kappa)$ from $(\mathfrak{A}, \mathsf{L}, G, \delta, \alpha)$ to $(\mathfrak{B}, \mathsf{L}, H, \delta, \beta)$ is a tunnel τ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and a ε-almost isometric isomorphism $(\varsigma_1, \varsigma_2)$ from (G, δ_G) to (H, δ_H).

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The *ε-reach* $\rho(\tau|\varepsilon)$ of a covariant tunnel $\tau = (\mathfrak{D}, \mathsf{L}, \pi_1, \pi_2, \varsigma_1, \varsigma_2)$ from $(\mathfrak{A}_1, \mathsf{L}_1, G_1, \delta_1, \alpha_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2, G_2, \delta_2, \alpha_2)$ is:

$$\max_{\{j,k\}=\{1,2\}} \sup_{\varphi \in \mathscr{S}(\mathfrak{A}_j)} \inf_{\psi \in \mathscr{S}(\mathfrak{A}_k)} \sup_{g \in G_j\left[\frac{1}{\varepsilon}\right]} \mathsf{mk}_{\mathsf{L}}(\varphi \circ \alpha_j^g \circ \pi_j, \psi \circ \alpha_k \circ \alpha_k^{\varsigma_k(g)} \circ \pi_k).$$

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The *covariant propinquity* $\Lambda^{\mathrm{cov}}(\mathbb{A}, \mathbb{B})$ between two Lipschitz dynamical systems $\mathbb{A}$ and $\mathbb{B}$ is:

$$\inf\left\{ \frac{\sqrt{2}}{2}, \varepsilon > 0 \middle| \exists \tau \text{ a } \varepsilon\text{-covariant tunnel } \mathbb{A} \to \mathbb{B} \text{ with } \max\{\chi(\tau), \rho(\tau|\varepsilon)\} \leqslant \varepsilon \right\}$$

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- a strongly continuous action of G on $\mathfrak{A}$ by Lipschitz morphisms.

Theorem (L., 18)

The *covariant propinquity* Λ^{cov} is a *metric* up to *equivariant* full quantum isometry, i.e it is symmetric, satisfies the triangle inequality, and:

$$\Lambda^{\mathrm{cov}}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, G, \delta_G, \alpha), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, H, \delta_H, \beta)) = 0$$

if and only if there exists a full quantum isometry $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \twoheadrightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ and an isometric isomorphism $\varsigma : G \to H$ such that:

$$\forall a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) \quad \forall g \in G \quad \pi \circ \alpha^g = \beta^{\varsigma(g)} \circ \pi.$$

Theorem (L., 17, 18)

If $D : [0,\infty) \to [0,\infty)$ and:

1. $\forall n \in \mathbb{N}$, $(\mathfrak{A}_n, \mathsf{L}_n, G_n, \delta_n, \alpha_n)$ is a Lipschitz dynamical F-system,
2. $\lim_{n\to\infty} \Upsilon((G_n,\delta_n),(G,\delta)) = 0$,
3. $\lim_{n\to\infty} \Lambda^*_F((\mathfrak{A}_n,\mathsf{L}_n),(\mathfrak{A},\mathsf{L})) = 0$,
4. for all $\varepsilon > 0$ there exists $\omega > 0, N \in \mathbb{N}$ such that:

$$\forall n \geqslant N \quad \forall g,h \in G_n \quad \delta_n(g,h) < \omega \implies \mathsf{mk}_{\mathsf{L}_n}(\alpha_n^g, \alpha_n^h) < \varepsilon,$$

then there exists an action α of G on $\mathfrak{A}$ and a $j : \mathbb{N} \to \mathbb{N}$ strictly increasing such that:

$$\lim_{n\to\infty} \Lambda^{\mathrm{cov}}_F((\mathfrak{A}_{j(n)}, \mathsf{L}_{j_n}, G_{j(n)}, \alpha_{j(n)}),(\mathfrak{A},\mathsf{L},G,\alpha)) = 0.$$

If G, G_n is a compact group and α_n *ergodic* $\forall_{\mathbb{N}} n$ then so is α.

Application to Closures

Theorem (L., 17)

Let $\mathscr{C}$ be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of $SU(2)$ by isometries. The *closure* of $\mathscr{C}$ for the dual propinquity consists only of quantum compact metric spaces over *type I* C*-algebras.

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Let $\mathscr{F}$ be the class of *finite dimensional quantum compact metric spaces* together with an ergodic action of $\mathbb{T}^2$ by isometries. The *closure* of $\mathscr{F}$ for the dual propinquity consists of fuzzy and quantum tori.

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Theorem (L., 18)

The *covariant propinquity* is *complete* on the class of Lipschitz dynamical systems with a bi-invariant metric.

Convergence of dual actions of quantum tori

Let ℓ be a *continuous length function* on $\mathbb{T}^d$. For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of $\widehat{G}$, for any $a \in C^*(\widehat{G},\sigma)$, set:

$$\mathsf{L}_{G,\sigma}(a) = \sup\left\{ \frac{\|\alpha^g_{G,\sigma}(a) - a\|_{C^*(\widehat{G},\sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where $\alpha_{G,\sigma}$ is the dual action of G on $C^*(\widehat{G},\sigma)$.

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Theorem (L., 18)

If $(G_n)_{n\in\mathbb{N}}$ is a sequence of closed subgroups of $\mathbb{T}^d$ converging to $\mathbb{T}^d$ for the Hausdorff distance Haus_ℓ and if σ_n are (lift of) multipliers of $\widehat{G_n}$ converging pointwise to some σ then:

$$\lim_{n\to\infty} \Lambda^{\mathrm{cov}}_F\Big(\big(C^*(\widehat{G_n},\eta), \mathsf{L}_{G_n,\eta}, G_n, \alpha_{G_n,\sigma_n}\big), \\ \big(C^*(\mathbb{Z}^d,\sigma), \mathsf{L}_{\mathbb{T}^d,\sigma}, \mathbb{T}^d, \alpha_{\mathbb{T}^d,\sigma}\big)\Big) = 0.$$

Completeness of certain classes of systems

An interesting complication arises when trying to determine when the covariant propinquity is complete. We introduce a notion of uniform equicontinuity.

Definition (L., 18)

Let $S = (G_n, \delta_n)_{n\in\mathbb{N}}$ be a sequence of proper monoids. The set of *regular sequences* $\mathscr{R}(S)$ is:

$$\left\{ (g_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G_n \,\middle|\, \begin{array}{l} \forall \varepsilon > 0 \quad \exists \omega > 0 \quad \exists N \in \mathbb{N} \\ \forall n \geqslant N \quad \forall h, k \in G_n \\ \delta_n(h,k) < \omega \implies \delta_n(hg_n, kg_n) < \varepsilon \end{array} \right\}.$$

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Theorem (L.,18)

$\mathscr{R}((G_n, \delta_n)_{n\in\mathbb{N}})$ is a monoid for the pointwise multiplication.

Complete classes for Υ

Theorem (L.,18)

Let $(G_n, \delta_n)_{n\in\mathbb{N}}$ be a equene of monoids, and $(\sum \varepsilon_n)_{n\in\mathbb{N}}$ be a convergent series of positive general term. If:

1. for all $n \in \mathbb{N}$, there exists a ε_n almost iso-iso $(\varsigma_n.\kappa_n)$ from G_n to G_{n+1},
2. for all $N \in \mathbb{N}$, with $r = \sum_{j=0}^{N} \varepsilon_j$, if $g \in G_N[r]$, then:

$$\left(\begin{cases} g_n = e_n \text{ if } n < N, \\ g_n = g \text{ if } n = N, \\ g_{n+1} = \varsigma_n(g_n) \text{ if } n \geqslant N \end{cases} \right)_{n\in\mathbb{N}} \in \mathscr{R}((G_n, \delta_n)_{n\in\mathbb{N}}),$$

then there exists a proper monoid (G, δ) such that:

$$\lim_{n\to\infty} \Upsilon\left((G_n, \delta_n), (G, \delta)\right) = 0.$$

Examples

Corollary (L.,18)

The space of proper monoids equipped with bi-invariant metrics is complete for the metric Υ.

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Corollary (L.,18)

If, for any proper monoids G and H, we set $\Upsilon'(G,H)$ as:

$$\left\{ \varepsilon > 0 \;\middle|\; \begin{array}{l} \exists(\varsigma,\kappa)\ \varepsilon \text{ iso-iso } G \to H \\ \sup_{g\in G} |\mathrm{dil}(h \mapsto hg) - \mathrm{dil}(h \mapsto h\varsigma(g))| < \varepsilon \\ \sup_{h\in H} |\mathrm{dil}(g \mapsto gh) - \mathrm{dil}(g \mapsto g\kappa(h))| < \varepsilon \end{array} \right\}$$

then Υ' is a complete extended metric.

Completeness for the covariant propinquity

Theorem (L.,18)

Let $(\mathfrak{A}_n, \mathsf{L}_n, G_m, \delta_n, \alpha_n)_{n\in\mathbb{N}}$ be a sequence of Lipschitz dynamical system, $R : [0,\infty) \to [0,\infty)$ continuous, and $(\sum \varepsilon_n)_{n\in\mathbb{N}}$ be a convergent series of positive terms such that:

1. $\Lambda((\mathfrak{A}_n, \mathsf{L}_n),(\mathfrak{A}_{n+1}, \mathsf{L}_{n+1})) \leqslant \varepsilon_n$,
2. (in particular) there exists (ς_n, κ_n) an ε_n-local almost isometry from G_n to G_{n+1} for all $n \in \mathbb{N}$,
3. $\forall n \in \mathbb{N}\ \forall g \in G_n \quad \mathsf{L}_n \circ \alpha_n^g \leqslant R(\delta_n(e_n, g))\mathsf{L}_n$,
4. $\forall N \in \mathbb{N}, g \in G_N \quad \left(e_0, \ldots, e_{N-1}, g, \varsigma_N(g), \ldots\right) \in \mathscr{R}((G_n, \delta_n)_{n\in\mathbb{N}})$,
5. $\forall \varepsilon > 0 \quad \exists \omega > 0, N \in \mathbb{N} \quad \forall n \geqslant N, \forall g, h \in G_n \quad \delta_n(g,h) < \omega \implies \mathsf{mk}_{\mathsf{L}}(\alpha_n^g, \alpha_n^h) < \varepsilon$

then there exists a Lipshitz dynamical system $(\mathfrak{A}, \mathsf{L}, G, \delta, \alpha)$ which is the limit of $(\mathfrak{A}_n, \mathsf{L}_n, G_n, \delta_n, \alpha_n)_{n\in\mathbb{N}}$ for the covariant propinquity.

Completeness

Corollary (L.,18)

Let $(\mathfrak{A}_n, \mathsf{L}_n, G_m, \delta_n, \alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of Lipschitz dynamical system for Λ^{cov}. Assume δ_n is bi-invariant for all $n \in \mathbb{N}$. Let $R : [0,\infty) \to [0,\infty)$ be continuous. If

1. $\forall n \in \mathbb{N}\ \forall g \in G_n \quad \mathsf{L}_n \circ \alpha_n^g \leqslant R(\delta_n(e_n, g))\mathsf{L}_n$,
2. $\forall \varepsilon > 0 \quad \exists \omega > 0, N \in \mathbb{N} \quad \forall n \geqslant N, \forall g, h \in G_n \quad \delta_n(g,h) < \omega \implies \mathsf{mk}_{\mathsf{L}}(\alpha_n^g, \alpha_n^h) < \varepsilon$

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Thank you!

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