Metric Spectral Triples on Inductive Limits of group C^* -algebras

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The Space of Noncommutative "Drums"

A. Connes introduced in 1985 a generalization of *spectral geometry* to the *noncommutative realm* by means of a structure called a *spectral triple*, i.e. a noncommutative analogue of the Dirac operators on Riemannian spin manifolds.

The Space of Noncommutative Manifolds

A *spectral triple* $\boxed{(\mathfrak{A}, \mathscr{H}, \not{D})}$ is an analogue of a first order pseudo-differential operator (of Dirac type), where

- $\mathfrak{A}$ is a unital C*-algebra, i.e. a noncommutative analogue of $C(X)$ for a compact Hausdorff space X,
- $\mathscr{H}$ is a Hilbert space on which $\mathfrak{A}$ acts, analogue to the space of sections of some spinor bundles,
- $\not{D}$ is a self-adjoint operator densely defined on $\mathscr{H}$ with compact resolvent and bounded commutator with a dense *-subalgebra of $\mathfrak{A}$.

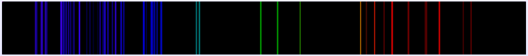

Figure: The spectrum of oxygen

The Space of Noncommutative Manifolds

Spectral triples are *much more flexible* than manifold structure, and can be constructed over *noncommutative C*-algebras*, fractals and other singular spaces, and even finite sets. Much success has been achieved in extending Atiah-Singer index theorem to spectral triples to many such generalized setting.

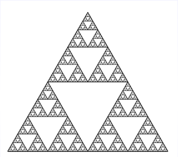

Figure: The Sierpiński Triangle

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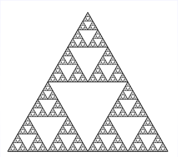

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We develop a geometric framework to study the *space of (metric) spectral triples* as a natural metric space to discuss problems from mathematical physics and functional analysis.

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- Matrix models: with $r_n = \exp\left(\frac{2i\pi}{n}\right)$, $e_1, \ldots, e_n$ canonical basis of $\mathbb{C}^n$ and if C_n and S_n are $n \times n$ matrices with $C_n e_j = r_n^{j-1} e_j$ and $S_n e_j = e_{(j-1 \mod n)}$, then $S_n C_n = r_n C_n S_n$; does $C^*(C_n, S_n)$ converge to $C(\mathbb{T}^2)$ as $n \to \infty$?

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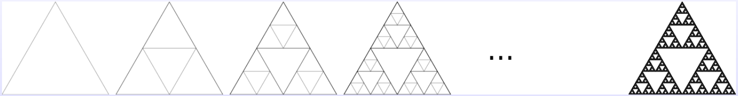
- Fractals and other spaces via approximations.



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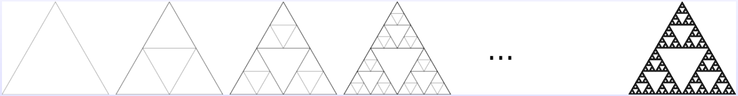

- Metric theory for noncommutative spaces:

$$\mathsf{mk}_{D\!\!\!/} : \varphi, \psi \in \mathscr{S}(\mathfrak{A}) \mapsto \sup\left\{ |\varphi(a) - \psi(a)| : |||[D\!\!\!/, a]|||_{\mathscr{H}} \leqslant 1 \right\}.$$

Past results

We define [L., 22] a *metric* up to *unitary equivalence* between *metric spectral triples* called the *spectral propinquity* Λ^{spec}, i.e.

$$\Lambda^{\mathrm{spec}}((\mathfrak{A}, \mathscr{H}, \mathbb{D}), (\mathfrak{B}, \mathscr{J}, \mathbb{P})) = 0$$

iff $\exists U$ unitary from $\mathscr{H}$ onto $\mathscr{J}$ with $U\mathrm{dom}(\mathbb{D}) = \mathrm{dom}(\mathbb{P})$,
$U^* \mathbb{D} U = \mathbb{P}$ and $a \in \mathfrak{A} \mapsto U^* a U$ a *-automorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.

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We proved that:

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The spectrum of spectral triples is continuous wrt spectral propinquity [L., 22], as is the continuous functional calculus in an appropriate sense.

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We will define the *Gromov-Hausdorff propinquity* and hint at the construction of the spectral propinquity.

Quantum compact Hausdorff spaces

Definition

A unital *C^*-algebra* $\mathfrak{A}$ is a unital associative algebra $\mathfrak{A}$ over $\mathbb{C}$ with a norm $\|\cdot\|_{\mathfrak{A}}$ such that:

1. $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$ is a Banach space,
2. $\forall a, b \in \mathfrak{A} \quad \|ab\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{A}}$,
3. there exists a conjugate linear, antimultiplicative involution $*$ on $\mathfrak{A}$ such that $\forall a \in \mathfrak{A} \quad \|a^* a\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}{}^2$.

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Theorem

A normed $*$-algebra $\mathfrak{A}$ is a C^*-algebra if, and only if $\mathfrak{A}$ is $*$-isomorphic to a closed self-adjoint algebra of bounded operators on a Hilbert space.

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- From the category of compact Hausdorff spaces: $X \mapsto C(X)$; $C(\pi : X \to Y) : f \in C(Y) \mapsto f \circ \pi \in C(X)$.

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Our work begins with a similar dual picture for *compact metric spaces*:

Founding Allegory of Noncommutative Geometry

Noncommutative geometry is the study of noncommutative generalizations of algebras of smooth functions over geometric spaces.

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Founding Allegory of Noncommutative Metric Geometry

Noncommutative *metric* geometry is the study of noncommutative generalizations of algebras of *Lipschitz* functions over *metric* spaces.

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\boxed{\mathsf{L}(f) = \sup\left\{\frac{|f(x) - f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y\right\}}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

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5. $\mathsf{L}\left(\frac{ab+ba}{2}\right) \vee \mathsf{L}\left(\frac{ab-ba}{2i}\right) \leqslant F\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}\right) + K\mathsf{L}(a)\mathsf{L}(b)$;

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We call L an *L-seminorm*.

Metric Spectral Triples

Definition (Connes, 85)

A $\boxed{\textit{spectral triple}}$ $(\mathfrak{A}, \mathscr{H}, D)$ is given by:

- a Hilbert space $\mathscr{H}$,
- a self-adjoint operator D defined on a sense subspace $\operatorname{dom}(D)$ of $\mathscr{H}$, with compact resolvent,
- a unital C*-algebra $\mathfrak{A}$, *-represented on $\mathscr{H}$,

such that

$$\mathfrak{A}_D = \{a \in \mathfrak{A} : a \operatorname{dom}(D) \subseteq \operatorname{dom}(D) \text{ and } [D, a] \text{ is bounded}\}$$

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is a dense *-subalgebra of $\mathfrak{A}$.
A spectral triple is *metric* when $(\mathfrak{A}, \mathsf{L}_{\not{D}})$ is a quantum compact metric space, where

$$\forall a \in \mathfrak{A}_{\not{D}} \cap \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{L}_{\not{D}}(a) := |||[\not{D}, a]|||_{\mathscr{H}}.$$

Examples of Metric Spectral Triples

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On the *noncommutative tori*, many constructions of metric spectral triples exist, including the usual flat Dirac operator and some of its perturbations, as well as limits of Dirac operators on fuzzy tori.

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On the *noncommutative tori*, many constructions of metric spectral triples exist, including the usual flat Dirac operator and some of its perturbations, as well as limits of Dirac operators on fuzzy tori.

Example

Certain spectral triples constructed over Podleś spheres and $SU_q(2)$ are metric (Kaad, Kyed,19-22).

Metric spectral triples from length functions

Let σ be a $\mathbb{T}$-valued 2-cocycle of a group G, i.e.

$$\forall x, y, z \in G \quad \sigma(x, y)\sigma(xy, z) = \sigma(xy, z)\sigma(y, z).$$

For each $g \in G$ and $\xi \in \ell^2(G)$, we define

$$\lambda(g)\xi : h \in G \mapsto \sigma(g, hg^{-1})\xi(hg^{-1}),$$

and we let $C^*_{\mathrm{red}}(G,\sigma) := C^*(\lambda(g) : g \in G\})$.
Let $\mathbb{L}$ be a length function over G, i.e. a function $\mathbb{L} : G \to [0,\infty)$ such that $\mathbb{L}(gg') \leqslant \mathbb{L}(g) + \mathbb{L}(g')$, $\mathbb{L}(g^{-1}) = \mathbb{L}(g)$ and $\{g \in G : \mathbb{L}(g) = 0\} = \{e\}$, for all $g, g' \in G$. Let $M_{\mathbb{L}}$ be the multiplication operator on $\ell^2(G)$.

Theorem (Connes, 89)

If $\mathbb{L}$ is *proper*, i.e. $\{g \in G : \mathbb{L}(g) \leqslant r\}$ is compact for all $r \geqslant 0$, then $(C^*_{\mathrm{red}}(G,\sigma), \ell^2(G), M_{\mathbb{L}})$ is a spectral triple.

Noncommutative Solenoids

A *noncommutative solenoid* $C^*(\mathbb{Z}[p^{-1}]^2, \sigma)$ is the twisted group C*-algebra of the group $\mathbb{Z}[p^{-1}]^2$ with p prime and

$$\mathbb{Z}[p^{-1}] := \left\{ \frac{m}{p^n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\} \subseteq \mathbb{Q}.$$

They were introduced and classified up to $*$-isomorphism, via their K-theory, by [L., Packer, 11].

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They were introduced and classified up to $*$-isomorphism, via their K-theory, by [L., Packer, 11].
Farsi, Landry and Packer (2022) introduced the length function:

$$\mathbb{L} : (g,h) \in \mathbb{Z}[p^{-1}]^2 \longmapsto |g| + \nu_p(g) + |h| + \nu_p(h)$$

where $\nu_p(g) := p^m$ where $g = p^m \frac{a}{b}$ where $\gcd(a,p) = \gcd(b,p) = 1$.
They prove that there exists a constant $C > 0$ such that for all $r \geqslant 1$:

$$\left|\left\{ g \in \mathbb{Z}[p^{-1}]^2 : \mathbb{L}(g) \leqslant 2r \right\}\right| \leqslant C \cdot \left|\left\{ g \in \mathbb{Z}[p^{-1}]^2 : \mathbb{L}(g) \leqslant r \right\}\right|.$$

By Christ-Rieffel (15), Long-Wu (19), this implies $(C^*(\mathbb{Z}[p^{-1}]^2, \sigma), \ell^2(\mathbb{Z}[p^{-1}]^2), M_{\mathbb{L}})$ is a metric spectral triple.

$\mathbb{Z}[p^{-1}]$

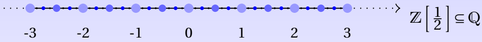

Figure: $\mathbb{Z}[p^{-1}]$ with $|\cdot|$

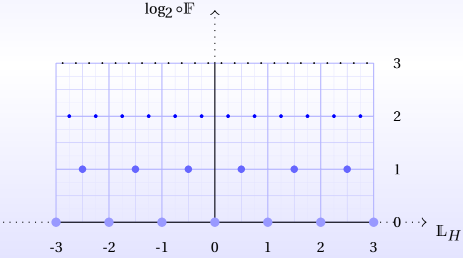

Figure: Another way to look at $\mathbb{Z}[p^{-1}]$

Inductive Limits

The noncommutative solenoids are inductive limits of quantum tori since:

$$\mathbb{Z}\left[\frac{1}{p}\right] = \bigcup_{n\in\mathbb{N}} \frac{1}{p^n}\mathbb{Z}.$$

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Our answer

We have defined a notion of convergence for spectral triples, which begins with the discovery of an analogue of the Gromov-Hausdorff distance for quantum compact metric spaces. We then devised an improved spectral triple for noncommutative solenoids which is motivated by this convergence question (and provides a positive answer).

GPS

The Gromov-Hausdorff Distance

Te *Hausdorff distance* Haus_d between two closed subsets A_1 and A_2 of a compact metric space (X, d) is defined by

$$\mathsf{Haus}_d(A_1, A_2) = \max_{\{j,k\}=\{1,2\}} \sup_{x \in A_j} \inf_{y \in A_k} d(x, y).$$

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Definition (Hausdorff, 1903; Edwards, 75; Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X,m_X) and (Y,m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X),\iota_Y(Y)) \;\middle|\; \begin{array}{l} (Z,\mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\},$$

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where Haus_d is the *Hausdorff distance* induced by a metric d.

The *Gromov-Hausdorff distance* is a *complete metric*, up to isometry, on the class of compact metric spaces.

Quantum Isometries

A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (99), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$ and

$$\forall b \in \operatorname{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\pi(\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})) = \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Quantum Isometries

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Theorem (Rieffel, 99)

If $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a quantum isometry, then $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is an isometry from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ into $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

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Theorem (L., 18)

If $(\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2)$ are two *unitarily equivalent metric spectral triples*, then $(\mathfrak{A}_1, \mathsf{L}_{\not{D}_1})$ and $(\mathfrak{A}_2, \mathsf{L}_{\not{D}_2})$ are *fully quantum isometric*.

The Dual Gromov-Hausdorff Propinquity

$$\begin{tikzcd}
& (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, "\pi_{\mathfrak{A}}"] \arrow[dr, "\pi_{\mathfrak{B}}"'] & \\
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Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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Definition (The extent of a tunnel, L. 13,14)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\}.$$

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The *dual propinquity* $\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.}$$

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Theorem (L., 13)

The *dual propinquity* Λ^*, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ by:

$$\boxed{\inf\left\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}}$$

is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

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In general, the propinquity allows one to talk about convergence of quantum compact metric spaces which are not subalgebras/quotients of each others. For instance, we have proven:

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3. Certain fractals, constructed as unions of curves, are limits, in the propinquity, of graphs, for the geodesic distance,
4. various families of AF algebras are continuous for an appropriate quantum metric structure: the function which maps the Baire space to UHF algebras is Lipschitz for the usual metric on the Baire space and the propinquity; the function which maps an irrational number in $(0, 1)$ to the Effros-Shen algebra for this number is continuous as well;

The situation for inductive limits

Theorem (Farsi,L.,Packer, 23)

Let $\mathfrak{A}_\infty = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be a unital C*-algebra, where $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of C*-subalgebras of $\mathfrak{A}_\infty$ with $1 \in \mathfrak{A}_0$.
For each $n \in \mathbb{N} \cup \{\infty\}$, let L_n be an L-seminorm on $\mathfrak{A}_n$. We assume:

$$\exists M > 0 \quad \forall n \in \mathbb{N} \quad \frac{1}{M}\mathsf{L}_n \leqslant \mathsf{L}_\infty \leqslant M\mathsf{L}_n \text{ on } \mathrm{dom}\left(\mathsf{L}_n\right).$$

The sequence $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ converges to $(\mathfrak{A}_\infty, \mathsf{L}_\infty)$ for the propinquity if, and only if, there exists a *-automorphism π of $\mathfrak{A}_\infty$ such that, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geqslant N$ then:

$$\forall a \in \mathrm{dom}\left(\mathsf{L}_\infty\right) \exists b \in \mathrm{dom}\left(\mathsf{L}_n\right) \quad \mathsf{L}_n(b) \leqslant \mathsf{L}_\infty(a) \text{ and } \|\pi(a) - b\|_{\mathfrak{A}_\infty} < \varepsilon \mathsf{L}_\infty(a),$$

and the similar property with the roles of a and b flipped.

Such a π is called a *bridge builder*.

Application to some group C-algebras*

Theorem (Farsi, L., Packer, 23)

Let $G_\infty = \bigcup_{n\in\mathbb{N}} G_n$ be a discrete group where $(G_n)_{n\in\mathbb{N}}$ is an increasing sequence of subgroups, and let σ be a 2-cocycle on G_∞.
Let $\mathbb{L}$ be a length function on G_∞ such that $\lim_{n\to\infty} \mathsf{Haus}_{\mathbb{L}}(G_n, G_\infty) = 0$.
Let γ_1, γ_2 be unitaries such that $\gamma_1^2 = \gamma_2^2 = 1$ and $\gamma_1\gamma_2 = -\gamma_2\gamma_1$, acting on a Hermitian space E.
Let $\mathbb{F} : g \in G_\infty \mapsto \mathrm{scale}(\min\{n \in \mathbb{N} : g \in G_n\})$ where $\mathrm{scale} : \mathbb{N} \to [0,\infty)$ is strictly increasing. If we define

$$\not{D} := M_{\mathbb{L}} \otimes \gamma_1 + M_{\mathbb{F}} \otimes \gamma_2$$

on $\left\{\xi \in \ell^2(G_\infty, E) : \sum_{g\in G_\infty} (\mathbb{L}(g)^2 + \mathbb{F}(g)^2) \left\|\xi(g)\right\|_E^2 < \infty\right\}$, and if $\not{D}_n := \not{D}_{|\ell^2(G_n,E)}$, then $(C^*_{\mathrm{red}}(G_n,\sigma), \ell^2(G_n)\otimes E, \not{D}_n)$ is a spectral triple.

A first convergence result

Theorem

Assume now that $(C^*(G_n,\sigma), \ell^2(G_n)\otimes E, \not{D}_n)$ is a *metric* spectral triple for all $n \in \mathbb{N}\cup\{\infty\}$. If

$$\{a \in \mathrm{dom}(\mathsf{L}_\infty) : \mathsf{L}_\infty(a) \leqslant 1\} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \{a \in \mathrm{dom}(\mathsf{L}_n) : \mathsf{L}_n(a) \leqslant 1\}\right)$$

then

$$\lim_{n\to\infty} \Lambda^*((C^*(G_n,\sigma), \mathsf{L}_{\not{D}_n}), (C^*(G_\infty,\sigma), \mathsf{L}_{\not{D}})) = 0.$$

In this proof, the identity is the bridge builder.

A natural question

What happens when a bridge builder π is also a full quantum isometry?

GPS

Tunnels between Spectral Triples

Let $(\mathfrak{A}_j, \mathscr{H}_j, \not{D}_j)$ be a metric spectral triple; set $\mathsf{L}_j = {\left|\kern-0.25ex\left|\kern-0.25ex\left| [\not{D}_j, \cdot] \right|\kern-0.25ex\right|\kern-0.25ex\right|}_{\mathscr{H}_j}$ and $\mathsf{D}_j(\xi) = \|\xi\|_{\mathscr{H}_j} + \left\|\not{D}_j \xi\right\|_{\mathscr{H}_j}$, $j \in \{1,2\}$.

$$\begin{tikzcd} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dl, bend right, "\theta_1"'] \arrow[dr, bend left, "\theta_2"] & \\ (\mathfrak{A}_1, \mathsf{L}_1) & & (\mathfrak{A}_2, \mathsf{L}_2) \end{tikzcd}$$

A tunnel: $\mathsf{L}_j(a) = \inf \mathsf{L}_{\mathfrak{D}}(\theta_j^{-1}(\{a\}))$.

Tunnels between Spectral Triples

Let $(\mathfrak{A}_j, \mathscr{H}_j, \not{D}_j)$ be a metric spectral triple; set $\mathsf{L}_j = \left|\!\left|\!\left| [\not{D}_j, \cdot] \right|\!\right|\!\right|_{\mathscr{H}_j}$ and $\mathsf{D}_j(\xi) = \|\xi\|_{\mathscr{H}_j} + \left\|\not{D}_j \xi\right\|_{\mathscr{H}_j}$, $j \in \{1,2\}$.

$$\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[d, dashed] \arrow[ddll, bend right, "\theta_1"'] \arrow[ddrr, bend left, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[dl, bend right, "\Pi_1"'] \arrow[dr, bend left, "\Pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{H}_1, \mathsf{D}_1) & & (\mathscr{H}_2, \mathsf{D}_2) & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"]
\end{tikzcd}$$

$\mathscr{J}$ is a $\mathfrak{D}$-module, $\mathsf{D}_j(\omega) = \inf \mathsf{T}(\Pi_j^{-1}(\{\omega\}))$, T D-norm

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$$
\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[ddd, "\theta_1"', bend right, to path={-- ([xshift=0]\tikztotarget)}] \arrow[dlll, phantom] \arrow[d, dashed] \arrow[dddll, "\theta_1"'] \arrow[dddrr, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[ddl, "\Pi_1"'] \arrow[ddr, "\Pi_2"] \arrow[d, dotted] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[ddl, "\pi_1"] \arrow[ddr, "\pi_2"'] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{H}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{H}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathbb{C}, 0) & & (\mathbb{C}, 0) &
\end{tikzcd}
$$

$\mathscr{J}$ is a $\mathfrak{D}$-$\mathfrak{C}$-C^*-corr; $(\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)$ tunnel.

Extent of Tunnels between Spectral Triples

Let $(\mathfrak{A}_j, \mathscr{H}_j, \not{D}_j)$ be a metric spectral triple; set $\mathsf{L}_j = \left|\left|\left|[\not{D}_j, \cdot]\right|\right|\right|_{\mathscr{H}_j}$ and $\mathsf{D}_j(\xi) = \|\xi\|_{\mathscr{H}_j} + \left\|\not{D}_j\xi\right\|_{\mathscr{H}_j}$, $j \in \{1, 2\}$.

$$
\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[dddll, "\theta_1"'] \arrow[dddrr, "\theta_2"] \arrow[d, dashed] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[ddl, "\Pi_1"'] \arrow[ddr, "\Pi_2"] \arrow[d, dotted] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{H}_1, \mathsf{D}_1) \arrow[dd, dotted] & & (\mathscr{H}_2, \mathsf{D}_2) \arrow[dd, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & & & & \\
 & (\mathbb{C}, 0) & & (\mathbb{C}, 0) &
\end{tikzcd}
$$

$$\chi(\tau) = \max\left\{\chi(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2), \chi(\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)\right\}.$$

Extent of Tunnels between Spectral Triples

We can generalize this picture to tunnels between any two metrical C*-correspondences.

$$\begin{tikzcd}
 & & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[d, dashed] \arrow[dddll, bend right, "\theta_1"'] \arrow[dddrr, bend left, "\theta_2"] & & \\
 & & (\mathscr{J}, \mathsf{T}) \arrow[ddl, bend right, "\Pi_1"'] \arrow[ddr, bend left, "\Pi_2"] & & \\
 & & (\mathfrak{C}, \mathsf{Q}) \arrow[u, dotted] \arrow[ddl, "\pi_1"] \arrow[ddr, "\pi_2"'] & & \\
(\mathfrak{A}_1, \mathsf{L}_1) \arrow[r, dashed, "\text{acts on}"'] & (\mathscr{M}_1, \mathsf{D}_1) \arrow[d, dotted] & & (\mathscr{M}_2, \mathsf{D}_2) \arrow[d, dotted] & (\mathfrak{A}_2, \mathsf{L}_2) \arrow[l, dashed, "\text{acts on}"] \\
 & (\mathfrak{B}, \mathsf{S}_1) & & (\mathfrak{B}_2, \mathsf{S}_2) &
\end{tikzcd}$$

$$\chi(\tau) = \max\left\{\chi\left((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2)\right), \chi\left((\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)\right)\right\}.$$

Covariant Reach of a Tunnel

Definition (L., 18)

Let $\tau = ((\mathscr{J}, \mathsf{T}, \ldots), (\Pi_1, \ldots), (\Pi_2, \ldots))$ be a metrical tunnel from $(\mathfrak{A}_1, \mathscr{H}_1, \mathbb{D}_1)$ to $(\mathfrak{A}_2, \mathscr{H}_2, \mathbb{D}_2)$. Let $\varepsilon > 0$. The *reach* $\rho^m(\tau|\varepsilon)$ of τ is

$$\underbrace{\sup_{\substack{\xi\in\mathscr{H}_j\\ \mathsf{D}_j(\xi)\leqslant 1}} \inf_{\substack{\eta\in\mathscr{H}_k\\ \mathsf{D}_k(\eta)\leqslant 1}}}_{\substack{\text{Hausdorff distance}\\ \forall\exists}} \overbrace{\sup_{0\leqslant t\leqslant\frac{1}{\varepsilon}}}^{\substack{\text{orbital}\\ \text{uniform}}} \underbrace{\sup_{\substack{\omega\in\mathscr{J}\\ \mathsf{T}(\omega)\leqslant 1}} \left| \langle \exp(it\mathbb{D}_j)\xi, \Pi_j(\omega)\rangle_{\mathscr{H}_j} - \langle \exp(it\mathbb{D}_k)\eta, \Pi_k(\omega)\rangle_{\mathscr{H}_k} \right|}_{\text{distance}}$$

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The *ε-magnitude* $\mu^m(\tau|\varepsilon)$ of τ is $\max\left\{\chi(\tau), \rho^m(\tau|\varepsilon)\right\}$.

The Spectral Propinquity

Definition (L., 18)

The *spectral propinquity* $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1),(\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2))$ between two metric spectral triples $(\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1)$ and $(\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2)$ is

$$\inf\left\{\frac{\sqrt{2}}{2},\varepsilon>0 : \exists\tau \text{ tunnel from } (\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1)\right. \\ \left.\text{to } (\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2) \text{ such that } \mu^m(\tau|\varepsilon)\leqslant\varepsilon\right\}.$$

The Spectral Propinquity

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$$\left. \text{to } (\mathfrak{A}_2,\mathscr{H}_2,\slashed{D}_2) \text{ such that } \mu^m(\tau|\varepsilon)\leqslant\varepsilon\right\}.$$

Theorem (L., 18)

The *spectral propinquity* Λ^{spec} is a *metric* on the class of spectral triples, up to unitary equivalence, i.e. $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1,\mathscr{H}_1,\slashed{D}_1),(\mathfrak{A}_2,\mathscr{H}_2,\slashed{D}_2))=0$ if, and only if there exists a *unitary* $U:\mathscr{H}_1\to\mathscr{H}_2$ such that $U\mathrm{dom}(\slashed{D}_1)=\mathrm{dom}(\slashed{D}_2)$,

$$U\slashed{D}_1U^*=\slashed{D}_2 \text{ and } \mathrm{Ad}_U \text{ *-isomorphism from } \mathfrak{A}_1 \text{ to } \mathfrak{A}_2.$$

GPS

Inductive Limits of Spectral Triples

Let $\mathfrak{A}_\infty = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be a unital C*-algebra, where $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of C*-subalgebras of $\mathfrak{A}_\infty$ with $1 \in \mathfrak{A}_0$. A spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$ is the *inductive limit* of the spectral triples $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ if $\mathscr{H}_\infty = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathscr{H}_n\right)$, with $(\mathscr{H}_n)_{n\in\mathbb{N}}$ an increasing sequence of Hilbert subspaces of $\mathscr{H}_\infty$ and such that $\mathfrak{A}_n\mathscr{H}_n \subseteq \mathscr{H}_n$ for all $n \in \mathbb{N}$; and where $\mathbb{D}_n = (\mathbb{D}_\infty)_{|\mathscr{H}_n}$.

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Theorem (Farsi, L., Packer, 23)

If $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \not{D}_\infty)$ is metric, if it is the inductive limit of metric spectral triples $(\mathfrak{A}_n, \mathscr{H}_n, \not{D}_n)_{n\in\mathbb{N}}$ as above, and if *there exists a bridge builder which is also a full quantum isometry*, then

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((\mathfrak{A}_n, \mathscr{H}_n, \not{D}_n), (\mathfrak{A}_\infty, \mathscr{H}_\infty, \not{D}_\infty)) = 0.$$

The main convergence result for dual of discrete groups

Theorem

Let $G_\infty = \bigcup_{n\in\mathbb{N}} G_n$ be a discrete group where $(G_n)_{n\in\mathbb{N}}$ is an increasing sequence of subgroups, and let σ be a 2-cocycle on G_∞.

Let $\mathbb{L}$ be a length function on G_∞ such that $\lim_{n\to\infty} \mathsf{Haus}_{\mathbb{L}}(G_n, G_\infty) = 0$.

Let $\mathbb{F} : g \in G_\infty \mapsto \mathrm{scale}(\min\{n\in\mathbb{N} : g \in G_n\})$ where $\mathrm{scale} : \mathbb{N} \to [0,\infty)$ is strictly increasing. We define

$$\not{D}_\infty := \begin{pmatrix} \mathbb{L}(g) & \mathbb{F}(g) \\ \mathbb{F}(g) & -\mathbb{L}(g) \end{pmatrix} \text{ and } \not{D}_n := \not{D}_{|\ell^2(G_n,\mathbb{C}^2)}.$$

If $(C^\ast(G_n,\sigma), \ell^2(G_n,\mathbb{C}^2), \not{D}_n)$ is a *metric* spectral triple for all $n \in \mathbb{N} \cup \{\infty\}$, and $\{a \in \mathrm{dom}(\mathsf{L}_\infty) : \mathsf{L}_\infty(a) \leqslant 1\} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \{a \in \mathrm{dom}(\mathsf{L}_n) : \mathsf{L}_n(a) \leqslant 1\}\right)$ then

$$\boxed{(C^\ast(G_n,\sigma), \ell^2(G_n,\mathbb{C}^2), \not{D}_n) \xrightarrow[\Lambda^{\mathrm{spec}}]{n\to\infty} (C^\ast(G_\infty,\sigma), \ell^2(G_\infty,\mathbb{C}^2), \not{D}_\infty).}$$

The main application

Let $G_\infty = \bigcup_{n\in\mathbb{N}} G_n$ be a group where $G_n \uparrow G_\infty$ are subgroups, $\mathbb{L}$ a length on G_∞ with $\mathsf{Haus}_{\mathbb{L}}(G_n, G_\infty) \xrightarrow{n\to\infty} 0$, and $\mathbb{F} : g \in G_\infty \mapsto f(\min\{n \in \mathbb{N} : g \in G_n\})$ with $f : \mathbb{N} \uparrow [0,\infty)$. Let

$$\not{D} := M_{\mathbb{L}} \otimes \gamma_1 + M_{\mathbb{F}} \otimes \gamma_2$$

on $\left\{\xi \in \ell^2(G_\infty, E) : \sum_{g\in G_\infty} (\mathbb{L}(g)^2 + \mathbb{F}(g)^2) \left\|\xi(g)\right\|_E^{\;2} < \infty\right\}$.

Theorem

Assume G_∞ is Abelian, and that $\mathbb{L}+\mathbb{F}$ is a length function such that

$$\exists C > 0 \ \forall r \geqslant 1 \quad |G_\infty[2r]| \leqslant C \cdot |G_\infty[r]|.$$

Then $(C^*(G_n, \sigma), \ell^2(G_n) \otimes E, \not{D}_n)$ is metric for all $n \in \mathbb{N}$, and

$$(C^*(G_n, \sigma), \ell^2(G_n) \otimes E, \not{D}_n) \xrightarrow[\Lambda^{\mathrm{spec}}]{n\to\infty} (C^*(G_\infty, \sigma), \ell^2(G_\infty) \otimes E, \not{D}_\infty).$$

Noncommutative Solenoids

We apply our result to $G_\infty := \mathbb{Z}\left[\frac{1}{p}\right]^d$, and for all $n \in \mathbb{N}$:

$$G_n := \left(\frac{1}{p^n}\mathbb{Z}\right)^d .$$

Fix σ any 2-cocycle of G_∞.
Let $\mathbb{L}$ be any norm on $\mathbb{R}^d$, so $\lim_{n\to\infty} \mathsf{Haus}_{\mathbb{L}}(G_n, G_\infty) = 0$. But $\mathbb{L}$ is not proper on G_∞, so we can not use Connes' spectral triple directly,
We set $\mathbb{F} : g \in G_\infty \mapsto p^{\min\{n\in\mathbb{N}: g \in G_n\}}$.
Then $\mathbb{L} + \mathbb{F}$ is proper, and even doubling!

$$\boxed{(C^*(\mathbb{Z}^d,\sigma), \ell^2(\mathbb{Z}^d)\otimes E, D\!\!\!\!/_{|n}) \xrightarrow[\Lambda^{\mathrm{spec}}]{n\to\infty} (C^*(\mathbb{Z}\left[p^{-1}\right]^d,\sigma), \ell^2(\mathbb{Z}[p^{-1}]^d)\otimes E, D\!\!\!\!/)}$$

Moreover, $\mathbb{F}$ is bounded on G_n, so we have found a spectral triple on noncommutative solenoids which is the limit of (bounded perturbations) of a standard metric spectral triple on quantum tori.

Bunce-Deddens Algebras

Let now $G_\infty := \mathbb{Z}(p^\infty) \times \mathbb{Z}$ where

$$\mathbb{Z}(p^\infty) := \left\{ z \in \mathbb{C} : \exists n \in \mathbb{N} \quad z^{(p^n)} = 1 \right\}.$$

Let $G_n := {}^{\mathbb{Z}}\!\big/_{p^n}$. Let $\mathbb{F} : g \in G_\infty \mapsto p^{\min\{n \in \mathbb{N} : g \in G_n\}}$. Let $\mathbb{L}(z, m) = \max\{|z - 1|, |m|\}$.

Then:

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}\left(\left(C^*\left({}^{Z}\!\big/_{p^n} \times \mathbb{Z}, \sigma\right), \ell^2\left({}^{\mathbb{Z}}\!\big/_{p^n} \times \mathbb{Z}\right) \otimes E, \mathbb{D}_n\right),\right.$$
$$\left.\left(C^*\left(\mathbb{Z}(p^\infty) \times \mathbb{Z}, \sigma\right), \ell^2\left(\mathbb{Z}(p^\infty) \times \mathbb{Z}\right) \otimes E, \mathbb{D}_\infty\right)\right) = 0.$$

For the right choice of cocycle, $C^*\left(\mathbb{Z}(p^\infty) \times \mathbb{Z}, \sigma\right)$ is the Bunce-Deddens algebra $\mathrm{BD}(p^\infty)$.

Thank you!

- *The Quantum Gromov-Hausdorff Propinquity*, F. Latrémolière, *Transactions of the AMS* **368** (2016) 1, pp. 365–411, ArXiv: 1302.4058
- *The Dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Journal de Mathématiques Pures et Appliquées* **103** (2015) 2, pp. 303–351, ArXiv: 1311.0104
- *A compactness theorem for the dual Gromov-Hausdorff Propinquity*, F. Latrémolière, *Indiana Univ. Math. J.* **66** (2017) 5, 1707–1753, Arxiv: 1501.06121
- *The modular Gromov-Hausdorff propinquity*, F. Latrémolière, Dissertationes Math. 544 (2019), 70 pp. 46L89 (46L30 58B34)
- *The Gromov-Hausdorff propinquity for metric Spectral Triples*, F. Latrémolière, *Adv. Math.* 404 (2022), paper no. 108393, arXiv:1811.04534
- *Metric approximations of spectral triples on the Sierpiński gasket and other fractal curves*, T. Landry, M. Lapidus, F. Latrémolière, *Adv. Math.* 385 (2021), Paper No. 107771, 43 pp, arXiv:2010.06921.
- *Convergence of Spectral Triples on Fuzzy Tori to Spectral Triples on Quantum Tori*, F. Latrémolière, *Comm. Math. Phys.* **388** (2021) 2, 1049–1128, arXiv:2102.03729.